DOMAINS OF τ-HOLOMORPHY ON A BANACH SPACE

BY

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TABLE OF CONTENTS

Page

ACKNOWLEDGEMENTS.................................... iii

ABSTRACT... v

INTRODUCTION....................................... 1

PRELIMINARIES...................................... 4

CHAPTER

I. SEQUENCES OF DOMAINS OF τ-HOLOMORPHY IN BANACH SPACES................................. 11

§1. Domains of τ-Holomorphy in a Complex Separable Banach Space.................. 11

§2. Sequences of Domains of τ-Holomorphy On a Complex and Separable Banach Space.... 28

§3. Sequences of Certain Domains on a Locally Convex Hausdorff Space.......... 40

II. KOBAYASHI AND CARATHEODORY DISTANCES FOR COMPLEX BANACH MANIFOLDS..................... 46

§1. Complex Analytic Banach Manifolds Over a Complex Banach Space.................... 46

§2. The Kobayashi Pseudodistance On a Complex Analytic Banach Manifold................. 50

§3. The Caratheodory Pseudodistance On a Complex Analytic Banach Manifold......... 57

BIBLIOGRAPHY....................................... 61

BIOGRAPHICAL SKETCH................................ 63

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Let E be a complex separable Banach space, U be a nonempty open subset of E, τ be a strictly positive lower semicontinuous function on U such that $d(x,\partial U) \geq \tau(x)$ for every $x \in U$. Let $\mathcal{B}_\tau(U)$ denote the collection of all finite unions of closed balls $\bar{B}_p(x)$ with center $x \in U$ and $p < \tau(x)$ and let $\mathcal{H}_\tau(U)$ be the algebra of the complex holomorphic functions in U which are bounded in every set of $\mathcal{B}_\tau(U)$ furnished with the Fréchet topology of the uniform convergence over the elements of $\mathcal{B}_\tau(U)$.

The main results are: if $A \in \mathcal{B}_\tau(U)$, then A is bounded, if and only if, its τ-holomorphy hull is bounded; if $A \in \mathcal{B}_\tau(U)$ and $A = \bigcup_{i=1}^{n} \bar{B}_{p_i}(x_i)$ and z_0 is some element of the τ-holomorphy hull of A, then, every $f \in \mathcal{H}_\tau(U)$ is holomorphic on $B_r(z_0)$ and for every $p < r$, $\| f\|_{B_p(z_0)} \leq \| f\|_{(p)_A}$, where $r = \min_{i=1,2,\ldots,n} \{\tau(x_i) - p_i\}$, $0 < p_i < \tau(x_i)$, $x_i \in U$; if U is bounded the following are equivalent:

(i) U is a domain of τ-holomorphy.

(ii) U is τ-holomorphically convex.

(iii) Every boundary point of U is τ-essential.

(iv) For every sequence $\{\xi_n\}_{n \in N}$ of points of U with $d(\xi_n, \partial U) \to 0$ as $n \to \infty$, there exists $f \in H_\tau(U)$ such that $\sup_{n \to \infty}\{|f(\xi_n)|\} = \infty$.

If U is bounded and is approximated from the inside by a principal sequence of domains $\{D_n\}_{n \in N}$, then if each domain D_n is τ-holomorphically convex relative to U, then U is a domain of τ-holomorphy; if in addition U is τ-semicomplete and if each of the domains D_n is τ-holomorphically convex relative to D_{n+1}, then U is domain of τ-holomorphy; if each D_n is a domain of τ-holomorphy and U is τ-complete, then U is a domain of τ-holomorphy.

Suppose now that E is a locally convex Hausdorff space and an open subset U of E is approximated from the inside by a principal sequence of domains $\{D_n\}_{n \in N}$. If each of the domains D_n is pseudoconvex, so is U; if each D_n is polynomially convex and if E has the approximation property, then U is pseudoconvex and Runge.

Finally it is shown that if M is a complex analytic Banach mainifold, then the Kobayashi pseudodistance is the largest for which every holomorphic mapping from the unit disk of the complex plane into a complex analytic Banach manifold is distance-decreasing while the Carathéodory pseudodistance is the smallest pseudodistance for which every holomorphic mapping from the complex analytic manifold to the unit disk of the complex plane is distance-decreasing.

INTRODUCTION

The study of holomorphic mappings defined on Banach spaces has received considerable attention in recent years. This dissertation contains new results relating to certain kinds of holomorphy on a complex Banach space.

The notion of a domain of holomorphy arises naturally with the study of holomorphic functions on a non-empty open subset of C^n. In particular, an open subset U of C^n is said to be a domain of holomorphy, if and only if, there exists a holomorphic function defined on it which cannot be extended analytically beyond any point of its boundary ∂U. It is known that in this case U is a domain of holomorphy, if and only if, U is holomorphically convex ([2], [10], [15], [17]).

The situation is different on complex Banach spaces. As a matter of fact, if U is a holomorphically convex domain in a complex Banach space, then it need not be a domain of holomorphy [11]. Moreover, the situation is different when one moves from one complex Banach space to another. The main reason is the behavior of the bounding sets on a complex Banach space. A closed bounded subset, A, of a complex Banach space, E, is said to be bounding if every complex valued holomorphic function on E is bounded on A. For certain complex Banach spaces, in particular, for separable or reflexive spaces, the bounding subsets are precisely the compact subsets, while there are examples of non-compact bounding subsets of other complex Banach spaces ([4], [5]).

In Sections 1 and 2 of Chapter I of this dissertation, we primarily consider a separable complex Banach space E and a kind of holomorphy, the τ-holomorphy, which is due to M. Matos. In particular, M. Matos has proved (see Preliminaries and Section I.1 for the corresponding notations and definitions):

THEOREM (MATOS). *Let E be a complex separable Banach space and let U be a non-empty open subset of E. Then, the following are equivalent:*

(a) U is a domain of τ-holomorphy.

(b) For every $A \in B_\tau(U)$, $\hat{A}_U^\tau$ is bounded and $d(\hat{A}_U^\tau, \partial U) > 0$.

(c) There is f in $H_\tau(U)$ such that it is impossible to find two open connected subsets U_1 and U_2 of E satisfying the following conditions:

(i) $U \cap U_1 \supset U_2$, $U_2 \neq \phi$, $U_1 \not\subset U$.

(ii) There is $f_1 \in H(U_1)$ such that $f = f_1$ on U_2.

In Section I.1, the primary result is a local continuation theorem, namely:

THEOREM I.1.10. *Let E be a complex separable Banach space. Let U be a non-empty open subset of E. Let $A \in B_\tau(U)$ and suppose that $A = \bigcup_{i=1}^{n} \overline{B}_{\rho_i}(x_i)$ and let $r = \min_{i=1,2,\ldots,n} \{\tau(x_i)-\rho_i\}$. Let z_0 be an element of $\hat{A}_U^\tau$. Then if $f \in H_\tau(U)$, then f is holomorphic on $B_r(z_0)$. And moreover, for every $p < r$ and for every $f \in H_\tau(U)$, we have $\|f\|_{B_p(z_0)} \leq \|f\|_{(p)_A}$.*

Section I.2 is concerned with the problem of convergence of a principal sequence of domains of τ-holomorphy. (See Section I.2 for the corresponding notations and definitions.) We are showing that:

THEOREM I.2.9. *Let E be a complex separable Banach space and let U be a bounded and τ-complete domain in E. Suppose that U is approximated from the inside by a principal sequence of domains of τ-holomorphy $\{D_\nu\}_{\nu\in N}$. Then U is a domain of τ-holomorphy.*

Section I.3 deals with the problem of convergence of certain kinds of domains on locally convex Hausdorff spaces.

In Chapter II, we introduce complex analytic Banach manifolds, and we furnish them with two pseudometrics, the Carathéodory and Kobayashi pseudodistances. We are also proving that the Kobayashi pseudodistance is the largest for which every holomorphic mapping from the unit disk of the complex plane into a complex analytic Banach manifold is distance-decreasing while the Carathéodory pseudodistance is the smallest pseudodistance for which every holomorphic mapping from the complex analytic manifold to the unit disk of the complex plane is distance-decreasing. The pseudodistances permit us to obtain main results on complex analytic Banach manifolds by a purely topological method. They enable us also to give geometric insight into function theoretic results.

In particular, for results in C^n see [3] and [12].

PRELIMINARIES

In this paper the notation and terminology used, unless otherwise stated, is that of Nachbin [16]. For the sake of completeness, however, we recall certain theorems and definitions needed in the sequence.

Let E and F be two complex Banach spaces. If $z \in E$ by $B_r(z)$ we denote the open ball with center z and of radius r while by $\bar{B}_r(z)$ we denote the closed ball of center z and of radius r. We reserve the letters R and C to denote the set of real and complex numbers respectively throughout this paper. Similarly, we reserve the letter N to denote the set of natural numbers.

THEOREM 0.1. *Let* $L^m(E,F) = \{T\colon T$ *is a continuous m-linear map from* E^m *into* $F\}$. *Then* $L^m(E,F)$ *is a Banach space with respect to the pointwise vector operations and norm defined by*

$$\|T\| = \sup_{x \in E^m} \frac{\|T(x_1, x_2, \ldots, x_m)\|}{\|x_1\| \ldots\ldots \|x_m\|} \quad \text{for } \|x_i\| \neq 0 \text{ and}$$

$i = 1, 2, \ldots, m$.

DEFINITION 0.2. *Let* $L_s^m(E,F)$ *denote the closed vector subspace of* $L^m(E,F)$ *of all m-linear maps* $T\colon E^m \to F$ *which are symmetric.*

DEFINITION 0.3. *Let* $A \in L^m(E,F)$. *Define its symmetrization* $A_s \in L_s^m(E,F)$ *by*

$$A_s(x_1,x_2,\dots,x_m) = (1/m!)\,\Sigma\, A(x_{j_1},x_{j_2},\dots,x_{j_m})$$

where the summation is taken over the m! permutations $(j_1,j_2,\dots,j_m)$ *of* $(1,2,\dots,m)$.

Note that $\|A_s\| \le \|A\|$ and moreover that the map $A \to A_s$ is a continuous projection from $L^m(E,F)$ onto $L_s^m(E,F)$. In the sequence we write $L^m(E)$ and $L_s^m(E)$ for $L^m(E,C)$ and $L_s^m(E,C)$ respectively. Finally, if $x \in E$ and $A \in L^m(E,F)$ we write Ax^m to denote $A(x,x,\dots,x)$ and Ax^0 to denote A.

DEFINITION 0.4. *A continuous m-homogeneous polynomial P from E into F is a mapping P for which there is a map* $A \in L^m(E,F)$ *such that* $P(x) = Ax^m$ *for every* $x \in E$. *We write* $P = \hat{A}$ *to denote that P corresponds to A that way.*

THEOREM 0.5. *If* $P^m(E,F)$ *denotes the set of all continuous m-homogeneous polynomials from E into F then,* $P^m(E,F)$ *becomes a Banach space with respect to the pointwise vector operations and norm defined by*

$$\|P\| = \sup_{x\neq 0} \frac{\|P(x)\|}{\|x\|} .$$

Observe that if m = 0 then $P^0(E,F)$ is just the set of all constant maps from E into F.

THEOREM 0.6. *The map* $A \to \hat{A}$ *from* $L_s^m(E,F)$ *onto* $P^m(E,F)$ *is a vector isomorphism and homeomorphism. Moreover*

$$\|\hat{A}\| \leq \|A\| \leq (m^m/m!)\,\|\hat{A}\|$$

and $(m^m/m!)$ *is the best universal constant.*

DEFINITION 0.7. *A continuous polynomial* P *from* E *into* F *is a mapping* $P: E \to F$ *for which there are* $m \in N \cup \{0\}$ *and* $P_k \in P^k(E,F)$ $(0 \leq k \leq m)$ *such that* $P = P_0 + P_1 + \ldots + P_m$. *If* $P \neq 0$ *there is one and only one such expression for* P *with* $P_k \neq 0$ *for some* $k = 0,1,2,\ldots,m$.

The degree of P *is defined to be the number* m. *If* $P = 0$ *then the degree of* P *may be taken as either equal to* -1 *or to* $-\infty$. *We denote by* $P(E,F)$ *the vector space of all continuous polynomials from* E *into* F.

DEFINITION 0.8. *A power series from* E *to* F *about* $x \in E$, *is a series in the variable* $z \in E$ *of the form*

$$(0.1) \qquad \sum_{m=0}^{\infty} A_m(z-x)^m = \sum_{m=0}^{\infty} P_m(z-x)$$

where $A_m \in L_s^m(E,F)$ *and* $P_m = \hat{A}_m$ *for every* $m = 0,1,2,\ldots$.

The radius of convergence of the above power series is the largest r, $r \in [0,\infty]$, *such that the power series is uniformly convergent on every* $B_p(x)$ *for* $0 \leq p < r$.

THEOREM 0.9. (Cauchy-Hadamard) *The radius of convergence of the power series of the definition 0.8. is given by*

$$r = (\limsup \|P_m\|^{1/m})^{-1}.$$

It is easy to verify that the power series of the Definition 0.8. is convergent, if and only if, the sequence $\{\| P_m \|^{1/m}\}_{m \in N}$ is bounded.

DEFINITION 0.10. *Let E and F be two complex Banach spaces and let U be a non-empty open subset of E. A mapping f from U into F is said to be holomorphic in U (in the Weierstrass sense) if corresponding to every* $x \in U$ *there are* $r > 0$ *and a power series of (0.1) from E to F about x such that*

(*i*) $B_r(x) \subset U$

(*ii*) $f(z) = \sum_{m=0}^{\infty} P_m(z-x)$ *uniformly for every* $z \in B_r(x)$.

In the case above we write $f \in H(U,F)$ and observe that $H(U,F)$ is a vector space with respect to the pointwise vector operations. We remark that the above series of Definition 0.10. is unique at every $x \in U$ and this series is called the Taylor's series of f at x.

Let $P_m \in P^m(E,F)$ correspond to $A_m \in L_s^m(E,F)$ by $P_m = \hat{A}_m$. We set the notations

$$d^m f(x) = m! A_m \text{ and } \hat{d}^m f(x) = m! \hat{A}_m$$

so we have the differential mappings

$$d^m f\colon x \in U \to d^m f(x) \in L_s^m(E,F)$$

$$\hat{d}^m f\colon x \in U \to \hat{d}^m f(x) \in P^m(E,F)$$

and the differential operators of order $m \in N \cup \{0\}$

$$d^m\colon f \in H(U,F) \to d^m f \in H(U,L_s^m(E,F))$$

$$\hat{d}^m\colon f \in H(U,F) \to \hat{d}^m f \in H(U,P^m(E,F)) \quad .$$

THEOREM 0.11. (Cauchy integral) *Let* $f \in H(U,F)$, $z \in U$, $x \in U$, *and* $r > 0$ *be such that* $(1-\lambda)z + \lambda x \in U$ *for every* $\lambda \in C$, *with* $|\lambda| \leq r$. *Then*

$$f(x) = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{f((1-\lambda)z+\lambda x)}{\lambda-1}\, d\lambda \quad .$$

THEOREM 0.12. *Let* $f \in H(U,F)$, $z \in U$, $x \in E$, *and* $r > 0$ *be such that* $z + \lambda x \in U$ *for every* $\lambda \in C$, *with* $|\lambda| \leq r$. *Then for every* $m \in N \cup \{0\}$ *we have*

$$(m!)^{-1}\, \hat{d}^m f(z)(x) = (2\pi i)^{-1} \int_{|\lambda|=r} (f(z+\lambda x)\lambda^{-(m+1)})\, d\lambda.$$

THEOREM 0.13. (Cauchy inequalities) *Let* E *and* F *be two complex Banach spaces and let* U *be a non-empty open subset of* E. *Suppose that* $f \in H(U,F)$ *and that for* $z \in U$ *there is some* $r > 0$ *such that* $\overline{B}_r(z)$ *is contained in* U. *Then for each* $m \in N \cup \{0\}$ *we have*

$$\left\| \frac{1}{m!}\, \hat{d}^m f(z) \right\| \leq \frac{1}{r^m} \sup_{\|x-z\|=r} \|f(x)\| \quad .$$

Some of the properties of the holomorphic mappings on C^n can be extended on the holomorphic mappings on a complex Banach space. We are proving here that the principle of analytic continuation is valid on complex Banach spaces.

THEOREM 0.14. *Let* E *and* F *be two complex Banach spaces and suppose that* f *is an holomorphic mapping from* E *into* F. *If* f *is equal to zero on some non-empty and open subset* U *of* E, *then we have* $f = 0$ *on* E.

PROOF: Let $S = \{z \in E: \hat{d}^m f(z) = 0 \text{ for all } m \in N \cup \{0\}\}$. Clearly S is non-empty. Moreover for fixed $m \in N$ the set $S_m = \{z \in E: \hat{d}^m f(z) = 0\}$ is closed because of the continuity of $\hat{d}^m f$ and then the set S is closed being the intersection of all closed sets S_m for $m \in N$. Now, let $a \in S$. Since f is assumed to be holomorphic on E and since $a \in S$ then, we can find some neighborhood V of a in E such that the Taylor's series expansion

$$\sum_{m=0}^{\infty} (m!)^{-1} \hat{d}^m f(a) \ (z-a)^m$$

converges to f(z) for every $z \in V$ according to the definition 0.10. But we have that $\hat{d}^m f(a) = 0$, because $a \in S$, for every $m \in N$. Hence we obtain that $f(z) = 0$ for every $z \in V$ and therefore $V \subset S$ and we can conclude that S is open. But because of connectedness we have that $E = S$ and then $f = 0$ on E.

COROLLARY 0.15. *Let E and F be two complex Banach spaces. Let $f \in H(E,F)$ and suppose that there exists some point $\alpha \in E$ such that $\hat{d}^m f(\alpha) = 0$ for every $m \in N \cup \{0\}$. Then $f = 0$ on E.*

COROLLARY 0.16. *Let E and F be two complex Banach spaces. Let $f, g \in H(E,F)$ and suppose that f and g agree on some open and non-empty subset of E. Then $f = g$ on E.*

DEFINITION 0.17. *Let U be a non-empty and open subset of a complex Banach space E. A mapping f from U into another* *complex Banach space* F *is said to be finitely holomorphic on* U *if the mapping* $f|U \cap S: U \cap S \to F$ *is holomorphic for every finite dimensional vector subspace* S *of* E.

THEOREM 0.18. (Nachbin) *Let* E *and* F *be two complex Banach spaces and let* U *be a non-empty open subset of* E. *A mapping* f *from* U *into* F *is finitely holomorphic on* U, *if and only if, for every pair of points* a *and* b *of* E *the mapping* $u: \{k \in C: a + bk \in U\} \to F$ *defined by* $u(k) = f(a + bk)$ *is holomorphic.*

THEOREM 0.19. (Nachbin) *Let* E *and* F *be two complex Banach spaces and let* U *be a non-empty and open subset of* E. *A mapping* f *from* U *into* F *is holomorphic, if and only if, it is finitely holomorphic and either it is continuous, or else it satisfies the equivalent conditions:*

(i) f *maps every compact subset of* U *onto a bounded subset of* F.

(ii) f *maps some neighborhood in* U *of every compact subset of* U *onto a bounded subset of* F.

We are closing this section by giving a new definition.

DEFINITION 0.20. *Let* E *be a complex Banach space and let* D *and* U *be two open and non-empty subsets of* E. *If* D *is a subset of* U *we will say that* D *is relatively compact in* U *if the closure of* D *is contained in* U *and is compact. In this case we write* $D \subset\subset U$.

CHAPTER I
SEQUENCES OF DOMAINS OF τ-HOLOMORPHY IN BANACH SPACES

§1. Domains of τ-Holomorphy in a Complex Separable Banach Space

Let E be a complex separable Banach space and let U be a non-empty open subset of E.

DEFINITION I.1.1. *A mapping f from U into C is said to be holomorphic if there exists a sequence of complex continuous n-homogeneous polynomials in E: $(n!)^{-1}\,\hat{d}^n f(x)$ for every $n \in N \cup \{0\}$ such that*

$$f(x+h) = \sum_{n=0}^{\infty} (n!)^{-1}\,\hat{d}^n f(x)\,(h)$$

converges uniformly for h in a neighborhood of zero in E.

Let ∂U denote the boundary of U and let τ be a strictly positive lower semicontinuous function of U such that $\tau(x) \leq d(x,\partial U)$ for every x in U.

Let $B_\tau(U)$ denote the collection of all finite unions of closed balls $\overline{B}_\rho(x)$ with center $x \in U$ and $\rho < \tau(x)$.

Let $H_\tau(U)$ denote the algebra of the complex holomorphic functions in U which are bounded in every set of $B_\tau(U)$ furnished with the Fréchet topology of the uniform convergence over the elements of $B_\tau(U)$ [14]. Observe that the union of $H_\tau(U)$, for all τ, is the algebra $H(U)$ of all the complex valued holomorphic mappings in U.

DEFINITION I.1.2. *Let E be a complex separable Banach space. Let U be a non-empty open subset of E, then U is said to be a domain of τ-holomorphy, if and only if, it is impossible to find two non-empty, open and connected subsets U_1 and U_2 of E satisfying the following conditions:*

(i) $U \cap U_1 \supset U_2$, $U_2 \neq \emptyset$, $U_1 \not\subset U$

(ii) for every $f \in H_\tau(U)$ there exists $F \in H(U_1)$ such that $F|U_2 = f$.

DEFINITION I.1.3. *Let E be a complex separable Banach space and let U be a non-empty open subset of E. If A is a non-empty subset of U we define the τ-holomorphy hull of A with respect to U to be the set*

$$\hat{A}_U^\tau = \{x \in U: |f(x)| \leq \|f\|_A \text{ for every } f \in H_\tau(U)\}$$

where $\|f\|_A = \sup_{x\in A}\{|f(x)|\}$.

From the Definition I.1.3. it follows that for every subset A of U we have $A \subset \hat{A}_U^\tau$ and therefore we can deduce

LEMMA I.1.4. *Let E be a complex separable Banach space and let U be a non-empty open subset of E. If A is a subset of U and if $\hat{A}_U^\tau$ is bounded, then A is bounded.*

If A is a subset of U we denote by $\overline{\text{co}}(A)$ the closed convex hull of A. Then the following theorem indicates that $\hat{A}_U^\tau$ is contained in the closed convex hull of A.

THEOREM I.1.5. *Let E be a complex separable Banach space and let U be a non-empty open subset of E. Then for every bounded subset A of U we have:* $\hat{A}^{\tau}_{U} \subset \overline{co}(A)$.

PROOF: Let $x_0 \notin \overline{co}(A)$. Then by Banach Separation Theorem [6] there exists a continuous linear functional f on E and real numbers c and δ, $\delta > 0$ such that

(I.1) $\operatorname{Re}\{f(\overline{co}(A))\} \leq c - \delta < c \leq \operatorname{Re}\{f(x_0)\}$.

Consider the set

$$P = \{\lambda \in C\colon \operatorname{Re}(\lambda) \leq \| \quad (f(x)) \|_{\overline{co}(A)}\}.$$

Then in view of the inequality (I.1) we obtain

(I.2) $f(\overline{co}(A)) \subset P$

and

(I.3) $f(x_0) \notin P$.

Using the facts that f is continuous and linear and in view of (I.2) and (I.3) we can find $\lambda_0 \in C$ and $r > 0$ such that

(I.4) $f(\overline{co}(A)) \subset B_r(\lambda_0)$

where $B_r(\lambda_0)$ denotes the open ball with center λ_0 and of radius r, and r can be chosen so that

(I.5) $f(x_0) \notin \overline{B}_r(\lambda_0)$.

We define a mapping ϕ from E into C by the rule $\phi(x) = f(x) - \lambda_0$. Then in view of (I.5)

(I.6) $|f(x_0) - \lambda_0| > r$

and in view of (I.4)

(I.7) $r \geq \sup_{x\in\overline{co}(A)} \{|f(x) - \lambda_0|\}.$

Hence in view of (I.6) and (I.7) we obtain

$$|\phi(x_0)| = |f(x_0) - \lambda_0| > r \geq \sup_{x\in\overline{co}(A)} \{|f(x) - \lambda_0|\} =$$

$$= \sup_{x\in\overline{co}(A)} \{|\phi(x)|\} > \sup_{x\in A}\{|\phi(x)|\} = \|\phi\|_A$$

and therefore

(I.8) $|\phi(x_0)| > \|\phi\|_A$.

Claim that $\phi \in \mathcal{H}_\tau(U)$.

Observe that ϕ is bounded on every subset of $\mathcal{B}_\tau(U)$ and therefore in order to establish the claim we must show that $\phi \in \mathcal{H}(U)$.

Let $\xi, \eta \in E$ and $S = \{z \in C: \xi + \eta z \in U\}$. Define a map u from S into C by the rule

$$u(\mu) = \phi(\xi+\mu\eta) \text{ for every } \mu \in S.$$

Then $u(\mu) = f(\xi) + \mu f(\eta) - \lambda_0$ and clearly then $u \in \mathcal{H}(S)$. Then ϕ is finitely holomorphic and since it is continuous it is holomorphic, by Theorem 0.19., on U and the claim is established. Finally, since $\phi \in \mathcal{H}_\tau(U)$ and because of inequality (I.8) we can deduce that $x_0 \notin \hat{A}_U^\tau$ and hence $\hat{A}_U^\tau \subset \overline{co}(A)$.

From Lemma I.1.4. and Theorem I.1.5. and the fact that if A is bounded, then $\overline{co}(A)$ is bounded, we obtain

THEOREM I.1.6. *Let E be a complex separable Banach space. Let U be a non-empty open subset of E. Suppose that* $A \subset U$*. Then A is bounded if and only if* $\hat{A}_U^\tau$ *is bounded.*

DEFINITION I.1.7. *Let E be a complex separable Banach space and U be a non-empty open and bounded subset of E. We will say that U is τ-holomorphically convex if and only if for every* $A \in B_\tau(U)$ *we have* $d(\hat{A}_U^\tau, \partial U) > 0$*.*

Then in the case of bounded open sets, Matos' theorem which is stated in the Introduction becomes:

THEOREM I.1.8. (Matos) *Let U be a non-empty open and bounded subset of a separable complex Banach space E. Then U is a domain of τ-holomorphy if and only if U is τ-holomorphically convex.*

LEMMA I.1.9. *Let U be a non-empty and open subset of a complex separable Banach space E. Let* $A \in B_\tau(U)$ *and suppose that there exists a constant* $M > 0$*, and a function* $f \in H_\tau(U)$ *such that* $|f(x)| \leq M$ *for every* $x \in A$*. Then the same estimate extends to* $\hat{A}_U^\tau$*. That is,* $|f(z)| \leq M$ *for every* $z \in \hat{A}_U^\tau$*.*

PROOF: Since $|f(x)| \leq M$ for every $x \in A$, we obtain $\|f\|_A \leq M$. But, since $z \in \hat{A}_U^\tau$, we have $|f(z)| \leq \|f\|_A$; and in view of the last inequality $|f(z)| \leq M$ for every $z \in \hat{A}_U^\tau$.

We are employing the following notation: Let U be a non-empty open subset of a separable complex Banach space E. Let $A \in B_\tau(U)$ and suppose that

$$A = \bigcup_{i=1}^{n} \overline{B}_{\rho_i}(x_i)$$

where $x_i \in U$ and $0 < \rho_i < \tau(x_i)$ for every $i = 1,2,\dots,n$. Let $r = \min_{i=1,\dots,n} \{\tau(x_i) - \rho_i\}$ and let $0 < q < r$. Then we denote by ${}^{(q)}A$ the set defined by

$${}^{(q)}A = \bigcup_{i=1}^{n} \overline{B}_{\rho_i+q}(x_i).$$

THEOREM I.1.10. *Let E be a complex separable Banach space. Let U be a non-empty open subset of E. Let $A \in B_\tau(U)$ and suppose that $A = \bigcup_{i=1}^{n} \overline{B}_{\rho_i}(x_i)$ and let $r = \min_{i=1,2,\dots,n} \{\tau(x_i)-\rho_i\}$. Let z_0 be an element of $\hat{A}_U^\tau$. Then, if $f \in H_\tau(U)$, then f is holomorphic on $B_r(z_0)$. And moreover, for every $p < r$ and for every $f \in H_\tau(U)$, we have*

$$\| f \|_{B_p(z_0)} \leq \| f \|_{{}^{(p)}A}.$$

PROOF: We first prove that if $f \in H_\tau(U)$, then f is holomorphic on $B_r(z_0)$.

Let $0 < q < r$ and let $\eta > 0$ such that $q - \eta > 0$. Consider the set ${}^{(q-\eta)}A$. Observe that ${}^{(q-\eta)}A \subset {}^{(r)}A$ and since every $f \in H_\tau(U)$ is bounded on ${}^{(r)}A$, then every $f \in H_\tau(U)$ is bounded on ${}^{(q-\eta)}A$. Let

$$\text{(I.9)} \quad \| f \|_{{}^{(q-\eta)}A} = M_f(\eta) \text{ for every } f \in H_\tau(U).$$

Now if z is any element of A, we clearly have that $B_{(q-\eta)}(z) \subset {}^{(q-\eta)}A$ and therefore in view of (I.9)

(I.10) $\|f\|_{B_{(q-\eta)}(z)} \leq M_f(\eta)$ for every $f \in H_\tau(U)$.

Now, since $H_\tau(U) \subset H(U)$, every $f \in H_\tau(U)$ belongs to $H(U)$, and therefore we can apply the Cauchy inequalities, of Theorem 0.13, to f and obtain:

(I.11) $|(j!)^{-1}\, \hat{d}^j f(z)| \leq M_f(\eta)\ (q-\eta)^{-j}$ for every $j = 0,1,\ldots$ and $z \in A$.

In view of Lemma I.1.9. and that $z_0 \in \hat{A}_U^\tau$, the above inequality becomes

(I.12) $|(j!)^{-1}\, \hat{d}^j f(z_0)| \leq M_f(\eta)\ (q-\eta)^{-j}$.

Now consider the power series of f at a neighborhood of z_0.

(I.13) $f(z) = \sum_{j=0}^{\infty} (j!)^{-1}\, \hat{d}^j f(z_0)\ (z-z_0)^j$.

Because of inequalities (10.4), we obtain:

(I.14) $|(j!)^{-1}\, \hat{d}^j f(z_0)\ (z-z_0)^j| \leq M_f(\eta)\ (|z-z_0|/(q-\eta))^j$.

But, (I.14) indicates that the series (I.13) is convergent for $|z-z_0| < q - \eta$. Hence, f is holomorphic on $B_{(q-\eta)}(z_0)$ which in turn implies that $f \in \mathcal{H}(B_q(z_0))$ because as η approaches zero, the series is convergent on $B_q(z_0)$. But q is arbitrary, $q < r$; and therefore, $f \in \mathcal{H}(B_r(z_0))$.

Now, we are proving the second part of the theorem. Let $0 < p < q - \eta$. Then, if $f \in \mathcal{H}_\tau(U)$, by the first part of the theorem we obtain that $f \in \mathcal{H}(B_p(z_0))$. Therefore, the mapping f admits a Taylor's series expansion.

$$\text{(I.15)} \quad f(z) = \sum_{j=0}^{\infty} (j!)^{-1} \hat{d}^j f(z_0) (z-z_0)^j \text{ for every } z \in B_p(z_0).$$

But $0 < p < q - \eta$ and therefore in view of (I.15) we obtain

$$|f(z)| \le \sum_{j=0}^{\infty} |(j!)^{-1} \hat{d}^j f(z_0) (z-z_0)^j| \le \sum_{j=0}^{\infty} M_f(\eta) (|z-z_0|/(q-\eta))^j$$
$$\le M_f(\eta) \sum_{j=0}^{\infty} (p/(q-\eta))^j$$

and therefore

$$\text{(I.16)} \quad |f(z)| \le M_f(\eta) \frac{1}{1 - \frac{p}{q-\eta}} \text{ for every } z \in B_p(z_0).$$

Define a function ϕ from R^+ to the R by

$$\text{(I.17)} \quad \phi(\eta) = (\| f\|_{B_p(z_0)}/M_f(\eta)).$$

We claim that $\phi(\eta) \le 1$ for all small η. Suppose not. Then there exists $\eta_0 > 0$ such that $\phi(\eta_0) > 1$. Consider the functions Φ_k from U into C defined for every $z \in U$ and $k \in N$ by

$$\text{(I.18)} \quad \Phi_k(z) = (f(z)/M_f(\eta_0))^k.$$

Clearly, $\Phi_k \in \mathcal{H}_\tau(U)$; and moreover,

$$\text{(I.19)} \quad \| \Phi_k\|_{B_p(z_0)} = (\phi(\eta_0))^k$$

because

$$\| \Phi_k\|_{B_p(z_0)} = \| (f(z)/M_f(\eta_0))^k\|_{B_p(z_0)} = (\| f\|_{B_p(z_0)}/M_f(\eta_0))^k =$$
$$= (\phi(\eta_0))^k.$$

In view of (I.19) and of the assumption that $\phi(\eta_0) > 1$, we can deduce that $\| \Phi_k\|_{B_p(z_0)}$ is large for large k. But,

$$\text{(I.20)}\quad M_{\Phi_k}(\eta_0) = 1$$

because

$$M_{\Phi_k}(\eta_0) = \|\Phi_k\|_{(q-\eta_0)_A} = \|(f(z)/M_f(\eta_0))^k\|_{(q-\eta_0)_A} =$$

$$= (M_f(\eta_0)/M_f(\eta_0))^k = 1.$$

Now since $\Phi_k \in \mathcal{H}_\tau(U)$, inequality (I.16) in view of (I.20) implies

$$\text{(I.21)}\quad \|\Phi_k\|_{B_p(z_0)} < \frac{1}{1 - \frac{p}{q-\eta}} .$$

But, the above inequality (I.21) implies that all functions $\{\Phi_k\}_{k=1}^{\infty}$ are uniformly bounded on $B_p(z_0)$ contradicting that Φ_k is large when k is large. Therefore, the claim is established. Since $\phi(\eta) \le 1$ for all small η, the equation (I.17) implies

$$(\|f\|_{B_p(z_0)}/M_f(\eta)) \le 1$$

or

$$\lim_{\eta\to q-p}(\|f\|_{B_p(z_0)}/M_f(\eta)) \le 1$$

or

$$\|f\|_{B_p(z_0)} \le M_f(q-p) = \|f\|_{q-(q-p)_A} = \|f\|_{(p)_A}$$

and finally from the above inequality, we obtain:

$$\|f\|_{B_p(z_0)} \le \|f\|_{(p)_A}.$$

THEOREM I.1.11. *Let E be a complex separable Banach space and let U be a non-empty open subset of E. Let Ω be a non-empty subset of U satisfying the property that every mapping $f \in H(U)$ is bounded in Ω. Then, there exists $A \in \mathcal{B}_\tau(U)$ such that $\hat{A}_U^\tau \supset \Omega$.*

PROOF: Suppose not. Let $A_1, A_2, \ldots, A_n, \ldots$ be a sequence of elements of $\mathcal{B}_\tau(U)$ such that every element $A \in \mathcal{B}_\tau(U)$ is contained in some A_j for some $j \in N$. Since $\Omega \not\subset \hat{A}_U^\tau$ for any $A \in \mathcal{B}_\tau(U)$, we obtain that $\Omega \not\subset \hat{A}_{j_U}^\tau$ for every $j \in N$.

Let $S_1 = A_1$. Clearly, $\Omega \not\subset \hat{S}_{1_U}^\tau$; and therefore, we can find an element ξ_1 of $\Omega - \hat{S}_{1_U}^\tau$. Let ρ_1 be such that $0 < \rho_1 < \tau(\xi_1)$ and define $S_2 = A_2 \cup B_{\rho_1}(\xi_1)$. Since $S_2 \in \mathcal{B}_\tau(U)$, we have that $\Omega \not\subset \hat{S}_{2_U}^\tau$; and therefore, we can find a new element $\xi_2 \in \Omega - \hat{S}_{2_U}^\tau$. Let ρ_2 be such that $0 < \rho_2 < \tau(\xi_2)$ and define S_3 by $S_3 = A_3 \cup B_{\rho_1}(\xi_1) \cup B_{\rho_2}(\xi_2)$.

Inductively then, we have obtained a sequence of subsets of U, $\{S_n\}_{n=1}^\infty$ and a sequence $\{\xi_n\}_{n=1}^\infty$ of points of Ω satisfying the properties:

(i) $\{S_n\}_{n=1}^\infty$ is a nested increasing sequence of sets.

(ii) $S_n \in \mathcal{B}_\tau(U)$ for every $n \in N$.

(iii) If $A \in \mathcal{B}_\tau(U)$, then there exists some $j \in N$ such that $A \subset S_j$.

(iv) $\xi_n \in S_j$ for every $j, n \in N$ with $n < j$.

(v) $\xi_n \notin \hat{S}_{n_U}^\tau$ for every $n \in N$.

Let $n \in \mathbb{N}$ and fixed. Claim that for every given positive number δ_n and every positive given number M_n we can find $f_n \in \mathcal{B}_\tau(U)$ such that

(I.22) $\quad |f_n(\xi_n)| = \delta_n \quad$ and $\quad \|f_n\|_{S_n} < M_n$

where ξ_n and S_n are as above.

Since $\xi_n \notin \hat{S}^{\tau}_{n_U}$ by the definition of the τ-holomorphy hull, we can find a mapping $g_n \in \mathcal{H}_\tau(U)$ such that $|g_n(\xi_n)| > \|g_n\|_{S_n}$. Then we take a mapping f_n from U into C defined by $f_n(x) = (g_n(x)/g_n(\xi_n))^m \delta_n$ for every $x \in U$ and some positive integer m. Then it is clear that $f_n \in \mathcal{H}_\tau(U)$; and moreover, for some large m the mapping f_n satisfies the conditions (I.22) and the claim is established.

We take $\delta_1 = 1$ and $M_1 = 1$. Apply (I.22) to get a mapping $f_1 \in \mathcal{H}_\tau(U)$ such that $|f_1(\xi_1)| = 1$ and $\|f_1\|_{S_1} < 1$.

Inductively, for

$$\delta_n = n + \sum_{i=1}^{n-1} |f_i(\xi_n)| \quad \text{and} \quad M_n = 2^{-(n-1)} \quad \text{for every } n \in \mathbb{N} - \{1\}$$

we can find according to (I.22) mappings $f_n \in \mathcal{H}_\tau(U)$ satisfying

$$|f_n(\xi_n)| = n + \sum_{i=1}^{n-1} |f_i(\xi_n)| \text{ and } \|f_n\|_{S_n} < 2^{-n} \text{ for } n \in \mathbb{N}-\{1\}.$$

By the construction of the functions $\{f_n\}_{n=1}^{\infty}$, we can see that the series $\sum_{i=1}^{\infty} f_i$ determines a function $f \in \mathcal{H}_\tau(U)$, since the series converges uniformly on each S_n and therefore on each $A \in \mathcal{B}_\tau(U)$. We claim that the function f is unbounded on Ω.

We have

$$\text{(I.23)} \quad |f(\xi_n)| = |f_n(\xi_n) - (f_n(\xi_n) + \sum_{i=1}^{\infty} f_i(\xi_n))| \geq$$

$$\geq |f_n(\xi_n)| - |\sum_{i=1}^{n-1} f_i(\xi_n) + \sum_{j=n+1}^{\infty} f_j(\xi_n)| \geq$$

$$\geq |f_n(\xi_n)| - \sum_{i=1}^{n-1} |f_i(\xi_n)| - \sum_{j=n+1}^{\infty} |f_j(\xi_n)|.$$

Taking into account conditions (I.22), inequalities (I.23) imply

$$|f(\xi_n)| \geq n - \sum_{j=n+1}^{\infty} |f_j(\xi_n)|.$$

But, recalling property (v), we have that whenever $j < n$ we have that $\xi_n \in S_j$. Then, $|f_j(\xi_n)| \leq \| f_j\|_{S_j} < 2^{-j}$. Hence, $\sum_{j=n+1}^{\infty} |f_j(\xi_n)| < 1$ and the above inequalities imply

$$|f(\xi_n)| > n-1.$$

But $\xi_n \in \Omega$, and therefore $\| f\|_{\Omega} = \infty$, and the claim is established. But every holomorphic map f in U was assumed to be bounded on Ω, and this is a contradiction.

DEFINITION I.1.12. *Let E be a complex separable Banach space, and let U be a non-empty open subset of E. A point z of the boundary of U is a τ-essential boundary point of U if there exists a mapping $f \in H_\tau(U)$ which is not the restriction of a holomorphic mapping in an open subset V of E containing z and satisfying $U \subset V$.*

The following theorem furnishes us with another necessary and sufficient condition in order for an open subset of a Banach space to be a domain of τ-holomorphy.

THEOREM I.1.13. *Let E be a separable complex Banach space. Let U be a non-empty open and bounded subset of E. Then U is a domain of τ-holomorphy, if and only if, every boundary point of U is τ-essential.*

PROOF: Observe that by the definition of a τ-essential boundary point it follows that even if U is not bounded, then U being a domain of τ-holomorphy, then every boundary point of U is τ-essential.

Conversely, suppose that every boundary point of U is τ-essential, but U is not a domain of τ-holomorphy. Then by the result of Theorem I.1.8., U is not τ-holomorphically convex. Therefore, we can find $A \in \mathcal{B}_\tau(U)$ such that

$$(I.24) \quad d(\hat{A}_U^\tau, \partial U) \nsucc 0.$$

Suppose that A is of the form

$$A = \bigcup_{i=1}^{n} \overline{B}_{\rho_i}(x_i),\ x_i \in U,\ 0 < \rho_i < \tau(x_i) \text{ for } i = 1,2,\dots,n.$$

Let $0 < r < \min_{1\le i\le n}\{\tau(x_i) - \rho_i\}$. Then in view of (I.24), we can find a point $z_0 \in \hat{A}_U^\tau$ satisfying

$$(I.25) \quad d(z_0, \partial U) < r.$$

But if $f \in \mathcal{H}_\tau(U)$, and since $z_0 \in \hat{A}_U^\tau$, by applying Theorem I.1.10. we deduce that $f \in \mathcal{H}(B_r(z_0))$. Combining this result with (I.25), we get $B_r(z_0) \cap U \neq \emptyset$. But every boundary point is assumed to be τ-essential and this is the contradiction.

The next theorem also furnishes us with another necessary and sufficient condition in order for an open subset of a complex separable Banach space to be a domain of τ-holomorphy.

THEOREM I.1.14. *Let E be a complex separable Banach space. Let U be a non-empty open and bounded subset of E. Then the following are equivalent:*

(i) U is a domain of τ-holomorphy.

(ii) For every sequence $\{\xi_n\}_{n=1}^{\infty}$ in U with $d(\xi_n, \partial U) \to 0$ as $n \to \infty$ then there exists a mapping $f \in H_\tau(U)$ such that $\sup_{n \to \infty}\{|f(\xi_n)|\} = \infty$.

PROOF: (i) → (ii). Since U is assumed to be a domain of τ-holomorphy, then U is τ-holomorphically convex by Theorem I.1.8. Therefore, for every $A \in B_\tau(U)$, we have that the set $\{\xi_n\} \not\subset \hat{A}_U^\tau$. But then Theorem I.1.11. supplies us with a mapping $f \in H_\tau(U)$ such that the condition (ii) is satisfied.

(ii) → (i). Suppose that U is not a domain of τ-holomorphy. Then there exists, by Theorem I.1.8. again, $A \in B_\tau(U)$ such that $d(\hat{A}_U^\tau, \partial U) = 0$. Let $\{\xi_n\}_{n\in N}$ be a sequence of points of $\hat{A}_U^\tau$ such that $d(\xi_n, \partial U) \to 0$ as $n \to \infty$. Let $f \in H_\tau(U)$ satisfying $\sup_{n \to \infty}\{|f(\xi_n)|\} = \infty$. Then, $\|f\|_{\hat{A}_U^\tau} = \infty$. But $f \in H_\tau(U)$, and from the definition of $\hat{A}_U^\tau$, it follows that we must have $\|f\|_{\hat{A}_U^\tau} = \|f\|_A < \infty$ and this is a contradiction.

THEOREM I.1.15. *Let E be a complex separable Banach space. Let U be a non-empty open and bounded subset of E. Then the following are equivalent:*

(*i*) *U is a domain of τ-holomorphy.*

(*ii*) *U is τ-holomorphically convex.*

(*iii*) *Every boundary point of U is τ-essential.*

(*iv*) *For every sequence $(\xi_n)_{n\in N}$ of points of U with $d(\xi_n, \partial U) \to 0$ as $n \to \infty$, there exists $f \in H_\tau(U)$ such that $\sup_{n\to\infty}\{|f(\xi_n)|\} = \infty$.*

The above Theorem is the collection of Theorems I.1.8., I.1.13. and I.1.14.

We are giving now some more properties of domains of τ-holomorphy.

THEOREM I.1.16. *Let E be a complex, separable Banach space. Let U be a non-empty, open subset of E. If U is a domain of τ-holomorphy, and ψ is a lower semicontinuous function defined on U and satisfying $\psi(x) \le \tau(x)$ for every $x \in U$, then U is a domain of ψ-holomorphy.*

PROOF: Let $A \in \mathcal{B}_\psi(U)$. Suppose that $A = \bigcup_{i=1}^{n} \overline{B}_{\rho_i}(x_i)$, where $\rho_i < \psi(x_i)$, $x_i \in U$ for each $i = 1,2,\ldots,n$.

By assumption, $\psi(x) \le \tau(x)$ for every $x \in U$; and therefore, $\rho_i \le \tau(x_i)$ for each $i = 1,2,\ldots,n$ which implies that $A \in \mathcal{B}_\tau(U)$. Claim

(I.25) $\hat{A}_U^\psi \subset \hat{A}_U^\tau$.

Let $z \notin \hat{A}_U^{\tau}$; then by the definition of the τ-holomorphy hull, we can find a mapping $F \in \mathcal{H}_\tau(U)$ such that $|F(z)| > \| F\|_A$. But $F \in \mathcal{H}_\tau(U)$, and $\psi < \tau$ implies that $F \in \mathcal{H}_\psi(U)$; and then in view of the above inequality involving the mapping F, we deduce that $z \notin \hat{A}_U^{\psi}$ and the claim is established.

Now since U is assumed to be a domain of τ-holomorphy, we obtain that $\hat{A}_U^{\tau}$ is bounded and that $d(\hat{A}_U^{\tau}, \partial U) > 0$. Then in view of (I.25), we obtain that $\hat{A}_U^{\psi}$ is bounded and that $d(\hat{A}_U^{\psi}, \partial U) > 0$ which indicates that U is a domain of ψ-holomorphy.

THEOREM I.1.17. *Let E be a complex separable Banach space. Let U and V be two non-empty open subsets of E with non-empty intersection. Suppose that U is a domain of τ-holomorphy and that V is a domain of ψ-holomorphy. Then $U \cap V$ is a domain of γ-holomorphy where $\gamma = \min(\tau|U \cap V, \psi|U \cap V)$.*

PROOF: Let $A \in \mathcal{B}_\gamma(U \cap V)$, and suppose that $A = \bigcup_{i=1}^{n} B_{\rho_i}(x_i)$, where $x_i \in U \cap V$ and $\rho_i < \gamma(x_i)$ for each $i = 1,2,\ldots,n$. But, $\gamma(x_i) \leq \min\{\tau(x_i), \psi(x_i)\}$ for each $i = 1,2,\ldots,n$, implies that $\rho_i < \tau(x_i)$ and $\rho_i < \psi(x_i)$ and therefore, $A \in \mathcal{B}_\tau(U)$ and $A \in \mathcal{B}_\tau(v)$. Claim:

(I.26) $\hat{A}_{U\cap V}^{\gamma} \subset \hat{A}_U^{\tau} \cap \hat{A}_V^{\psi}$.

We recall that $\hat{A}_{U\cap V}^{\gamma} = \{z \in U \cap V: |f(z)| \leq \| f\|_A$ for every $f \in \mathcal{H}_\gamma(U \cap V)\}$. Let $z \notin \hat{A}_U^{\tau} \cap \hat{A}_V^{\psi}$. Then, we may suppose that either $z \in \hat{A}_U^{\tau}$ and $z \notin \hat{A}_V^{\psi}$, or that $z \notin \hat{A}_U^{\tau}$ and $z \notin \hat{A}_V^{\psi}$. But, in both cases, since $z \notin \hat{A}_V^{\psi}$, we can find some mapping $f \in \mathcal{H}_\psi(V)$ such that $|f(z)| > \| f\|_A$. But, $f \in \mathcal{H}_\psi(V)$ and $\gamma \leq \psi$ on $U \cap V$

implies that $f \in H_{\gamma}(U \cap V)$ which combined with the above inequality gives us that $z \notin \hat{A}^{\gamma}_{U\cap V}$. Hence, in either case above, (I.26) is established.

Using the fact that U is a domain of τ-holomorphy, we obtain:

$$\text{(I.27)} \quad d(\hat{A}^{\gamma}_{U\cap V}, \partial U) \geq d(\hat{A}^{\tau}_{U}, \partial U) > 0,$$

and the fact that V is a domain of ψ-holomorphy, we obtain:

$$\text{(I.28)} \quad d(\hat{A}^{\gamma}_{U\cap V}, \partial V) \geq d(\hat{A}^{\psi}_{V}, \partial V) > 0.$$

Then (I.27) and (I.28) imply that

$$\text{(I.29)} \quad d(\hat{A}^{\gamma}_{U\cap V}, \partial U \cup \partial V) > 0$$

and then (I.29) with the aid of the property $\partial U \cup \partial V \supset \partial(U \cup V)$ gives:

$$\text{(I.30)} \quad d(\hat{A}^{\gamma}_{U\cap V}, \partial(U \cap V) > 0.$$

Finally, V being a domain of τ-holomorphy implies that $\hat{A}^{\tau}_{U}$ is bounded; and similarly, V being a domain of ψ-holomorphy implies that $\hat{A}^{\psi}_{V}$ is bounded. Then, because of (I.26), $\hat{A}^{\gamma}_{U\cap V}$ is bounded. Then the above result, together with (I.30), proves the Theorem.

If S is a bounding set in a domain of τ-holomorphy V on a separable complex Banach space, then S is closed in E by Theorem I.1.11. Then, by a result of S. Dineen [5], S is compact. Hence, all bounding sets of a domain of τ-holomorphy are compact. We note here that the fact E is a complex, separable Banach space is critical in this case. In fact, if

E is not separable, there exist bounding sets in U which are not compact. An example is given in [5]. Specifically, let $E = \ell_\infty$ (the space of all bounded sequences furnished with the sup norm topology) and $A = \bigcup_{n \in N} u_n$ where $u_n = (0,\ldots,0,1,0,0,\ldots)$, the 1 appears in the nth place. Then A is a bounding set in ℓ_∞, but not compact.

We are closing this chapter by stating two more theorems. The first of which is an immediate result of Theorem I.1.17 and the second of which is due to M. Matos.

THEOREM I.1.19. *Let E be a complex separable Banach space. Let $\{V_\nu\}_{\nu=1}^{n}$ be a finite collection of open sets in E. Suppose that V_ν is a domain of τ_ν-holomorphy for each $\nu = 1,2,\ldots,n$. Let Ω denote the intersection of all sets V_ν. If Ω is non-empty, then it is a domain of γ-holomorphy where $\gamma = \min\{\tau_1|\Omega, \tau_2|\Omega, \ldots, \tau_n|\Omega\}$.*

THEOREM I.1.20. *Let E be a complex separable Banach space and let V be a non-empty open subset of E. Then, the following are equivalent.*

(i) V is a domain of τ-holomorphy.

(ii) For every closed subset S of V, S is bounded and $d(S,\partial V) > 0$ if every $f \in H_\tau(V)$ is bounded on S.

§2. Sequences of Domains of τ-Holomorphy on a Complex and Separable Banach Space

In the sequence a domain U in a complex separable Banach space will be a non-empty, open, and connected subset U of E. Also, the meaning of the function τ wherever it appears will be the same as in §1. That is, τ will denote a strictly positive lower semicontinuous function defined on a domain U in a complex separable Banach space E; and for each $x \in U$ it satisfies $d(x,\partial U) \geq \tau(x)$. Finally, if D is some domain in E, which is a subset of a domain U in E, by $H_\tau(D)$ we will denote the collection of all holomorphic functions from D into C which are bounded on every $A \in B_\tau(U)$ which is contained in D.

DEFINITION I.2.1. *Let E be a complex Banach space. Let U be a domain in E and let $\{D_\nu\}_{\nu \in N}$ be a sequence of domains in E. We will say that the sequence of domains $\{D_\nu\}_{\nu \in N}$ approximates U; and we will write $\lim\limits_{\nu\to\infty} D_\nu = U$, if for every $z \in U$ there exists an open neighborhood V_z of z, subset of U, which lies in the intersection of almost all domains of the sequence $\{D_\nu\}_{\nu \in N}$.*

Observe that the above definition is equivalent to $\lim\limits_{\nu\to\infty} D_\nu = U$ if and only if $z \notin U$, then z lies inside the intersection of finite many of the domains of the sequence $\{D_\nu\}_{\nu \in N}$.

DEFINITION I.2.2. *Let E be a complex Banach space, U be a domain in E, and $\{D_\nu\}_{\nu \in N}$ be a sequence of domains in E. If for each $\nu \in N$, we have that $D_\nu \subset\subset D_{\nu+1} \subset\subset U$; and if $\lim\limits_{\nu\to\infty} D_\nu = U$, then we will say that the sequence $\{D_\nu\}_{\nu \in N}$ is a principal sequence of domains approximating U from the inside.*

DEFINITION I.2.3. *Let E be a complex separable Banach space, and let U_1 and U_2 be two non-empty open subsets of E. Suppose that in addition $U_1 \subset U_2$. Let A be a non-empty subset of U_1.*

We define the τ-holomorphy hull of A relative to U_2 to be the set $\hat{A}^{\tau}_{1,2}$, where

$$\hat{A}^{\tau}_{1,2} = \{z \in U_1 : |f(z)| \le \|f\|_A, \text{ for every } f \in H_\tau(U_2)\}$$

and where τ is defined on U_2.

DEFINITION I.2.4. *Let E be a complex separable Banach space, and let U_1 and U_2 be two non-empty open subsets of E. Suppose that in addition $U_1 \subset U_2$. Let $A \in B_\tau(U_2)$ such that $A \subset U_1$. We will say that U_1 is τ-holomorphically convex relative to U_2, if for every such A we have*

(*i*) $\hat{A}^{\tau}_{1,2}$ *is bounded.*

(*ii*) $d(\hat{A}^{\tau}_{1,2}, \partial U_1) > 0$.

THEOREM I.2.5. *Let E be a complex, separable Banach space and let U be a bounded domain in E. Suppose that $\{D_\nu\}_{\nu \in N}$ is a principal sequence of domains in E approximating U from the inside. If for every $\nu \in N$, each domain D_ν is τ-holomorphically convex relative to U; then U is a domain of τ-holomorphy.*

PROOF: Suppose that U is not a domain of τ-holomorphy. Then according to the Theorem I.1.8., U is not τ-holomorphically convex. Therefore, we can find some $A \in B_\tau(U)$ such that $d(\hat{A}^{\tau}_U, \partial U) \not> 0$ where $A = \bigcup_{i=1}^{n} \overline{B}_{\rho_i}(x_i)$, $x_i \in U$ and $\rho_i < \tau(x_i)$ for each $i = 1,2,\ldots,n$. Let p and r be two positive real numbers satisfying $0 < p < r < \min_{1 \le i \le n}\{\tau(x_i) - \rho_i\}$. Since $d(\hat{A}^{\tau}_U, \partial U) \not> 0$, if $z_0 \in \partial U$, we can find a sequence of elements of $\hat{A}^{\tau}_U$, say $\{z_\kappa\}_{\kappa \in N}$, such that $\lim_{\kappa \to \infty} z_\kappa = z_0$. Since each $z_\kappa \in \hat{A}^{\tau}_U$, we obtain

(I.31) $|f(z_\kappa)| \leq \|f\|_A$, for each $\kappa = 1,2,\dots,n$ and $f \in \mathcal{H}_\tau(U)$.

But then according to the Theorem I.1.10., we have that

(I.32) $\|f\|_{B_p(z_\kappa)} \leq \|f\|_{{}^{(p)}A}$ for each $\kappa = 1,2,\dots,n$ and $f \in \mathcal{H}_\tau(U)$.

Using the fact that z_0 is the limit of the sequence $\{z_\kappa\}$ as κ approaches infinity, inequality (I.32) yields

(I.33) $\|f\|_{B_p(z_0)} \leq \|f\|_{{}^{(p)}A}$ for each $f \in \mathcal{H}_\tau(U)$.

By our hypothesis, the given sequence $\{D_\nu\}_{\nu \in N}$ is a principal sequence of domains approximating U from the inside. But we have that ${}^{(p)}A \subset U$, and therefore, we can find some natural number μ such that ${}^{(p)}A \subset D_\nu$ for every $\nu \geq \mu$.

By our assumption again each domain D_ν is τ-holomorphically convex with respect to U; and therefore, in view of the above remark we have that

(I.34) $d({}^{(p)}\hat{A}^\tau_{\nu,\infty}, \partial D_\nu) > 0$, for every $\nu \geq \mu$

where ${}^{(p)}\hat{A}^\tau_{\nu,\infty}$ denotes the τ-holomorphy hull of ${}^{(p)}A$ relative to U. Moreover, from Definition I.2.3. we have that

(I.35) $|f(z)| > \|f\|_{{}^{(p)}A}$ for $z \in (D_\nu - {}^{(p)}\hat{A}^\tau_{\nu,\infty})$ and $f \in \mathcal{H}_\tau(U)$.

Now, $z_0 \in \partial U$ and $\lim\limits_{\nu\to\infty} D_\nu = U$ imply that we can find some natural number λ such that

(I.36) $\partial D_\nu \cap B_p(z_0) \neq \emptyset$ for every $\nu \geq \lambda$.

Let $\theta = \max\{\mu,\lambda\}$. Then both inequalities (I.34) and (I.35) are valid simultaneously for $\nu \geq \theta$. That is,

(I.37) $d({}^{(p)}\hat{A}^{\tau}_{j,\infty}, \partial D_j) > 0$ and $\partial D_j \cap B_p(z_0) \neq \emptyset$ for $j \geq \theta$.

But the above inequality implies that if z^* is an element of $(D_j - {}^{(p)}\hat{A}^{\tau}_{j,\infty})$, then for sufficiently large j we will have that $z^* \in B_p(z_0)$. Combining the above result with inequality (I.35), we obtain that for sufficiently large j we have

(I.39) $|f(z^*)| > \|f\|_{{}^{(p)}A}$, for every $f \in H_\tau(U)$.

But then the above found inequality (I.39) contradicts the inequality (I.33). Hence U is τ-holomorphically convex and therefore U is a domain of τ-holomorphy.

DEFINITION I.2.6. *Let E be a complex separable Banach space and let U be a domain in E. Suppose that D_1, D_2 and D_3 are three domains in U satisfying $D_1 \subset D_2 \subset D_3$. Suppose, furthermore, that for every $A \in B_\tau(U)$ such that $A \subset D_1$, we have that $\hat{A}^{\tau}_{1,3}$ is bounded and that $d(\hat{A}^{\tau}_{1,3}, \partial D_2) > 0$. We will say that the domain U is τ-semicomplete if for every given $\varepsilon > 0$ and every $f \in H_\tau(D_2)$ we can find a function $F \in H_\tau(D_3)$ such that $\|F-f\|_{D_1} < \varepsilon$.*

We are proving:

THEOREM I.2.7. *Let E be a complex separable Banach space. Let U be a bounded τ-semicomplete domain in E. Suppose that $\{D_\nu\}_{\nu\in N}$ is a principal sequence of domains approximating U*

from the inside. If each $\mathcal{D}_\nu$ is τ-holomorphically convex relative to $\mathcal{D}_{\nu+1}$, for each $\nu \in N$, then U is a domain of τ-holomorphy.

PROOF: In view of the Theorem I.2.5., we have to show that each of the domains D_ν is τ-holomorphically convex relative to U. That is, if $A \in \mathcal{B}_\tau(U)$ and $A \subset D_\nu$ then $d(\hat{A}^\tau_{\nu,\infty}, \partial D_\nu) > 0$ where

$$\hat{A}^\tau_{\nu,\infty} = \{z \in D_\nu\colon |f(z)| \le \|f\|_A \text{ for every } f \in \mathcal{H}_\tau(U)\}.$$

Equivalently, since each D_ν is τ-holomorphically convex relative to $D_{\nu+1}$, we have that $d(\hat{A}^\tau_{\nu,\nu+1}, \partial D_\nu) > 0$. Hence, it is enough to show that $\hat{A}^\tau_{\nu,\infty} \subset \hat{A}^\tau_{\nu,\nu+1}$, or that

$$\text{(I.40)} \quad (D_\nu - \hat{A}^\tau_{\nu,\nu+1}) \subset (D_\nu - \hat{A}^\tau_{\nu,\infty}).$$

Let $\zeta \in (D_\nu - \hat{A}^\tau_{\nu,\nu+1}) = (D_\nu - \{z \in D_\nu\colon |f(z)| \le \|f\|_A$ for $f \in \mathcal{H}_\tau(D_{\nu+1})\})$. Then we can find a function $f_0 \in \mathcal{H}_\tau(D_{\nu+1})$ such that $|f_0(\zeta)| > \|f_0\|_A$. Let δ be a positive real number such that

$$\text{(I.41)} \quad |f_0(\zeta)| > \|f_0\|_A + \delta.$$

We select a sequence of positive real numbers, $\{\varepsilon_\rho\}_{\rho\in N}$, such that $\sum_{\rho\in N} \varepsilon_\rho < (\delta/2)$.

Now we have that $D_\nu \subset D_{\nu+1} \subset D_{\nu+2}$, and since each D_ν is τ-holomorphically convex relative to $D_{\nu+1}$, and the latter is τ-holomorphically convex relative to $D_{\nu+2}$, we have

$d(\hat{A}^{\tau}_{\nu,\nu+2}, \partial D_{\nu+1}) > 0$, and since U is τ-semicomplete from Definition I.2.6., it follows that we can find a $f_1 \in H_\tau(D_{\nu+2})$ such that

$$(I.42)\quad |f_0(z) - f_1(z)| < \varepsilon_1 \text{ for every } z \in D_\nu.$$

Inductively we obtain a sequence $\{f_\kappa\}_{\kappa \in N}$ satisfying:

(i) $f_\kappa \in H_\tau(D_{\nu+\kappa+1})$ for each $\kappa \in N$,

(ii) $|f_{\rho_1}(z) - f_{\rho_2}(z)| < \sum_{\lambda=\rho_2+1}^{\rho_1} \varepsilon_\lambda$ for each $\kappa > \nu$,

$\rho_1 > \rho_2 > \kappa - \nu$, where the last inequality was obtained from

$$|f_{\rho_1}(z) - f_{\rho_2}(z)| = |\sum_{\lambda=\rho_2+1}^{\rho_1} (f_\lambda(z) - f_{\lambda-1}(z)| \le$$

$$\le \sum_{\lambda=\rho_2+1}^{\rho_1} |f_\lambda(z) - f_{\lambda-1}(z)| < \sum_{\lambda=\rho_2+1}^{\rho_1} \varepsilon_\lambda .$$

Observe that the inequalities in the property (ii) above imply that the sequence $\{f_\nu(z)\}_{\nu=0}^{\infty}$ is a Cauchy sequence, and therefore, it determines a function $f \in H_\tau(U)$. Moreover, the function f satisfies for every $z \in D_\nu$ the following inequality

$$(I.43)\quad |f_0(z) - f(z)| < (\delta/2)$$

because

$$|f_0(z) - f(z)| = \lim_{\rho\to\infty} |f_0(z) - f_\rho(z)|$$

and in view of the property (ii) above the latter equality yields

$$|f_0(z) - f(z)| \le \sum_{\rho=1}^{\infty} |f_\rho(z) - f_{\rho-1}(z)| < \sum_{\rho=1}^{\infty} \varepsilon_\rho < (\delta/2).$$

We claim that

(I.44) $|f(\zeta)| > \| f\|_A$.

We obtain from inequality (I.43)

(I.45) $|f_0(z)| - |f(z)| < (\delta/2)$ for every $z \in D_\nu$.

The above inequality for $z = \zeta$ yields

(I.46) $|f_0(\zeta)| - |f(\zeta)| < (\delta/2)$

or

(I.47) $|f(\zeta)| + (\delta/2) > |f_0(\zeta)|$.

Combining the above inequality with inequality (I.41) we obtain

(I.48) $|f(\zeta)| + (\delta/2) > \| f_0\|_A + \delta$.

From (I.43) again we obtain

$$|f(z)| - (\delta/2) < |f_0(z)| \text{ for every } z \in D_\nu$$

or

(I.49) $\| f\|_A - (\delta/2) \le \| f_0\|_A$.

Combining (I.48) and (I.49) we obtain

$$|f(\zeta)| + (\delta/2) > \| f_0\|_A + \delta \ge \| f\|_A - (\delta/2) + \delta = \| f\|_A + (\delta/2)$$

and hence

$$|f(\zeta)| > \| f\|_A.$$

Therefore, we have shown that if $\zeta \in (D_\nu - \hat{A}^\tau_{\nu,\nu+1})$ then $\zeta \in (D_\nu - \hat{A}^\tau_{\nu,\infty})$ and (I.40) is established.

DEFINITION I.2.8. *Let E be a complex separable Banach space and let U be a bounded τ-semicomplete domain in E. Let* $\mathcal{D}_1$, $\mathcal{D}_2$, *and* $\mathcal{D}_3$ *be three domains such that* $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_3 \subset U$. *We will say that U is a τ-complete domain, if and only if, for every* $A \in B_\tau(U)$ *with* $A \subset \mathcal{D}_1$ *and* $d(\hat{A}^\tau_{1,3}, \partial\mathcal{D}_2) > 0$, *then we can find a domain of τ-holomorphy* Ω_1 *such that* $\mathcal{D}_1 \subset \Omega_1 \subset \mathcal{D}_2$.

THEOREM I.2.9. *Let E be a complex separable Banach space and let U be a bounded and τ-complete domain in E. Suppose that U is approximated from the inside by a principal sequence of domains of τ-holomorphy* $\{\mathcal{D}_\nu\}_{\nu\in N}$. *Then U is a domain of τ-holomorphy.*

PROOF: Let for every $\mu = 1,2,\ldots,$ and $\nu = 0,1,2,\ldots,$ $M_{\mu,\nu}$ and $m_{\mu,\nu}$ be two real numbers satisfying for each μ and ν:

$$M_{\mu,\nu} = \max_{q\in\partial D_\mu}\{d_{\mu\nu}(q)\},\quad m_{\mu,\nu} = \min_{q\in\partial D_\mu}\{d_{\mu\nu}(q)\}$$

where by $d_{\mu\nu}(q)$ we denote the distance of the point $q \in \partial D_\mu$ from the boundary of D_ν and by $d_{\mu o}(q)$ we have denoted the distance of the point $q \in \partial D_\mu$ from the boundary of U.

Now, from the sequence of domains $\{D_\nu\}_{\nu\in N}$ we extract a subsequence of domains $\{D_{\nu_\rho}\}_{\rho\in N}$ in the following manner:

(i) Choose $D_{\nu_1} = D_1$.

(ii) D_{ν_2} is so chosen so that $M_{\nu_2,0} < m_{\nu_1,0}$.

(iii) D_{ν_3} is so chosen so that $M_{\nu_2,\nu_3} < m_{\nu_1,\nu_3}$ and that

$$M_{\nu_{3,0}} < m_{\nu_{2,0}}.$$

It is clear that the above constructed sequence $\{D_{\nu_\rho}\}_{\rho\in N}$ is a principal sequence of domains of τ-holomorphy and in general the domain $D_{\nu_{\rho+1}}$, for $\rho > 1$, has been chosen so that

(I.50) $M_{\nu_\rho,\nu_{\rho+1}} < m_{\nu_{\rho-1},\nu_{\rho+1}}$ for every $\rho = 2,3,\ldots,$

and

(I.51) $M_{\nu_{\rho+1,0}} < m_{\nu_{\rho,0}}$ for every $\rho = 2,3,\ldots$.

We claim that if $A \in \mathcal{B}_\tau(U)$ and $A \subset D_{\nu_{\rho-1}}$ then

(I.52) $d(\hat{A}^\tau_{\nu_{\rho-1},\nu_{\rho+1}}, \partial D_{\nu_\rho}) > 0.$

If not, then we can find a point $z_0 \in \partial D_{\nu_\rho}$ and a sequence of points $\{z_\kappa\}_{\kappa=1}^\infty$ of $\hat{A}^\tau_{\nu_{\rho-1},\nu_{\rho+1}}$ such that $\lim_{\kappa\to\infty} z_\kappa = z_0$. But since

$\hat{A}^\tau_{\nu_{\rho-1},\nu_{\rho+1}} = \{z \in D_{\nu_{\rho-1}} : |f(z)| \leq \|f\|_A$, for every $f \in \mathcal{H}_\tau(D_{\nu_{\rho+1}})\} \subset \{z \in D_{\nu_{\rho+1}} : |f(z)| \leq \|f\|_A$, for every $f \in \mathcal{H}_\tau(D_{\nu_{\rho+1}})\} = \hat{A}^\tau_{D_{\nu_{\rho+1}}}$ we obtain that $z_\kappa \in \hat{A}^\tau_{D_{\nu_{\rho+1}}}$.

Let $r = m_{\nu_{\rho-1},\nu_{\rho+1}}$; then ${}^{(r)}A \in \mathcal{B}_\tau(U)$ and clearly ${}^{(r)}A \subset D_{\nu_{\rho+1}}$. Then appealing to the Theorem I.1.10., we obtain that every mapping $f \in \mathcal{H}_\tau(D_{\nu_{\rho+1}})$ is holomorphic on $B_r(z_\kappa)$. But $\lim_{\kappa\to\infty} z_\kappa = z_0$, and therefore for every $f \in \mathcal{H}_\tau(D_{\nu_{\rho+1}})$ we have, $f \in \mathcal{H}_\tau(B_r(z_0))$. But in view of (I.50) we have that $r > M_{\nu_\rho,\nu_{\rho+1}}$. But $D_{\nu_{\rho+1}}$ was assumed to be a domain of τ-holomorphy, and the latter inequality leads to a contradiction. Hence (I.52) has been established.

By assumption, U is τ-complete and therefore in view of (I.52), we can find a domain of τ-holomorphy Ω_ρ for each $\rho = 2,3,\ldots$, such that

$$D_{\nu_{\rho-1}} \subset \Omega_\rho \subset D_{\nu_\rho} .$$

Because of the above inclusion, we can infer that $\lim_{\rho\to\infty}\Omega_\rho = U$ where $\{\Omega_\rho\}_{\rho\geq 2}$ is a principal sequence of domains of τ-holomorphy which are approximating U from the inside.

We now claim that each domain Ω_ρ is τ-holomorphically convex relative to $\Omega_{\rho+1}$ for every $\rho = 2,3,\ldots$. Let $A \in \mathcal{B}_\tau(U)$ such that $A \subset \Omega_\rho$. We must show that

$$\text{(I.53)} \quad \hat{A}^\tau_{\Omega_\rho,\Omega_{\rho+1}} \subset \hat{A}^\tau_{\Omega_\rho}$$

because, then, since Ω_ρ is a domain of τ-holomorphy, we will have that $d(\hat{A}^\tau_{\Omega_\rho}, \partial\Omega_\rho) > 0$, and then (I.53) will imply that $d(\hat{A}^\tau_{\Omega_\rho,\Omega_{\rho+1}}, \partial\Omega_\rho) > 0$; and therefore this will establish that Ω_ρ is τ-holomorphically convex relative to $\Omega_{\rho+1}$.

Let $z \in \hat{A}^{\tau}_{\Omega_\rho,\Omega_{\rho+1}}$. Then $z \in \Omega_\rho$ and $|f(z)| \le \|f\|_A$ for every $f \in H_\tau(\Omega_{\rho+1})$. If now $F \in H_\tau(\Omega_\rho)$, then, since U is τ-complete, we can find some $G \in H_\tau(\Omega_{\rho+1})$ such that

(I.54) $\|G - F\|_{\Omega_\rho} < (\varepsilon/2)$.

For $z \in \Omega_\rho$, the above inequality implies

(I.55) $|F(z)| - |G(z)| < (\varepsilon/2)$,

(I.56) $|F(z)| < (\varepsilon/2) + |G(z)|$.

On the other hand, we have that $z \in \hat{A}^{\tau}_{\Omega_\rho,\Omega_{\rho+1}}$, $G|\Omega_\rho \in H_\tau(\Omega_\rho)$, and hence

(I.57) $|G(z)| \le \|G\|_A$.

Then combining (I.56) and (I.57) we obtain

(I.58) $|F(z)| < (\varepsilon/2) + \|G\|_A$.

Appealing one more time to (I.54) we obtain

(I.59) $\|G\|_A < (\varepsilon/2) + \|F\|_A$.

Finally, combining the last two inequalities, we obtain

(I.60) $|F(z)| < \varepsilon + \|F\|_A$.

But ε is arbitrary and also F is an arbitrary element of $H_\tau(\Omega_\rho)$. Therefore (I.60) establishes (I.53).

Hence, we have exhibited a principal sequence of domains $\{\Omega_\rho\}_{\rho>1}$ approximating U from the inside; and moreover, each domain Ω_ρ is τ-holomorphically convex relative to $\Omega_{\rho+1}$ for every $\rho = 2,3,\ldots$. Then, the above conclusion together with Theorems I.2.5. and I.2.7. completes the proof.

§3. Sequences of Certain Domains On a Locally Convex Hausdorff Space

Let E be a locally convex Hausdorff space over C, and let cs(E) denote the set of all continuous seminorms on E. For $\alpha \in cs(E)$, $x \in E$ and $r > 0$ the "α-ball" about x of radius r is defined to be the set

$$B_r^\alpha(x) = \{y \in E: \alpha(x-y) < r\}.$$

The "α-boundary distance" $d_U^\alpha: U \to [0,\infty]$ for an open non-empty subset U of E, is defined for all $x \in U$ by

$$d_U^\alpha(x) = \sup\{r > 0: B_r^\alpha(x) \subset U\}.$$

For a subset K of U we put

$$d_U^\alpha(K) = \inf\{d_U^\alpha(x): x \in K\}.$$

Another distance function $\delta_U: U\times E \to [0,\infty]$ is given for all pairs $(x,a) \in U\times E$ by

$$\delta_U(x,a) = \sup\{r > 0: x + \lambda a \in U \text{ for all } \lambda \in C \text{ with } |\lambda| < r\}.$$

DEFINITION I.3.1. *Let E be a locally convex Hausdorff space. An open non-empty subset U of E is said to be pseudoconvex if the function $-\log\delta_U$ is plurisubharmonic.*

Let $P(U)$ denote the set of all plurisubharmonic functions on U; let $H(U)$ denote the vector space of all holomorphic functions on U; and let $\pi(E)$ denote the space of all continuous polynomials from E into C.

For $Q \subset P(U)$ and $K \subset U$, we define as the "Q-convex hull" of K to be the set $\hat{K}_Q$ defined by

$$\hat{K}_Q = \{x \in U: v(x) \leq \sup v(y),\ v \in Q,\ y \in K\}.$$

For $A \subset H(U)$ and $K \subset U$ we define as the "A-holomorphy convex hull" of K to be the set $\hat{K}_A$ defined by

$$\hat{K}_A = \{x \in U: |f(x)| \leq \|f\|_K,\ f \in A\}.$$

DEFINITION I.3.2. *Let E be a locally convex Hausdorff space. An open non-empty subset U of E is called holomorphically convex if $\hat{K}_{H(U)}$ is precompact in U for every compact and non-empty subset K of U.*

A subset K of U will be called precompact here, if it is relatively compact; and moreover, if there exists $\alpha \in cs(E)$ such that $d_U^\alpha(K) > 0$.

We note here that a holomorphically convex open set $U \subset E$ is pseudoconvex. The converse is true for $E = C^n$, for $E = C^{(IN)}$ [9], for C^A [1], and for certain Banach spaces E which are separable and have basis [18]. It is an open question whether the converse holds in general.

DEFINITION I.3.3. *Let E be a locally convex Hausdorff space. An open non-empty subset U of E is said to be polynomially* *convex if $\hat{K}_{\pi(E)}$ is precompact in U for all compact and non-empty subsets K of U.*

DEFINITION I.3.4. *Let E be a locally convex Hausdorff space. An open non-empty subset U of E is said to be Runge if $\pi(E)$ is dense in $H(U)$ with respect to the compact open topology. Then U will be called finitely Runge (respectively finitely polynomially convex) if for every finite dimensional vector subspace F of E, $U \cap F$ is Runge (respectively polynomially convex) in F.*

THEOREM I.3.5. *Let E be a locally convex Hausdorff space. Let U be a non-empty open subset of E approximated from the inside by a principal sequence of domains $\{D_n\}_{n\in N}$. If each of the domains D_n is Runge, so is U.*

PROOF: Suppose that U is not Runge. Then according to the Theorem I.3.4., we can find a function $f \in H(U)$, a compact subset K of U such that

$$(I.61) \quad |f(x) - p(x)| \geq \varepsilon \quad \text{for every } p \in \pi(E) \text{ and } x \in K$$

where ε is some given positive real number.

Now for each $x \in K$, we select an open set D_x of the given sequence D_n which contains x. Then the collection of all such selected domains D_x forms an open cover for K. But K being compact admits a finite subcover, say $\{D_{x_i}\}_{i=1}^{m}$. Let j be the maximum of the numbers $x_1, x_2, \ldots, x_m$. Then since the given sequence is principal, we obtain that the domain D_j covers K.

But $f \in H(U)$, so its restriction F on D_j is holomorphic there; and then in view of (I.61), we obtain that D_j is not Runge and this is absurd.

THEOREM I.3.6. *Let E be a locally convex Hausdorff space. Let U be a non-empty open and connected subset U of E. Suppose that $\{D_n\}_{n\in N}$ is a sequence of pseudoconvex domains in E approximating U from the inside. If $D_n \subset D_{n+1}$, for every $n \in N$, then U is pseudoconvex.*

PROOF: Consider the distance functions $\delta_{D_n} : D_n\times E \to [0,\infty]$, for each $n \in N$, and the distance function $\delta_U : U\times E \to [0,\infty]$. Clearly, from the definition of δ_{D_n} and the fact that the sequence of the domains $\{D_n\}_{n\in N}$ is nested, it follows that if $x \in U$, then there exists a natural number n_x such that for all $n \geq n_x$ and $a \in E$ we have

$$\delta_{D_n}(x,a) \leq \delta_{D_{n+1}}(x,a), \quad \delta_{D_n}(x,a) \leq \delta_U(x,a)$$

or by taking logarithms we can infer that

$$\text{(I.62)} \quad -\log\delta_{D_n}(x,a) \geq -\log\delta_{D_{n+1}}(x,a) \geq -\log\delta_U(x,a)$$

where this holds for every $n \geq n_x$ and $(x,a) \in U\times E$.

According to the Definition I.3.1. and the fact that each domain D_n is assumed to be pseudoconvex, we obtain that each function $-\log\delta_{D_n}$ is plurisubharmonic on $D_n\times E$ for each $n \in N$. Hence, the restrictions of the functions $-\log\delta_{D_n}(x,a)$ on every complex line of $U\times E$, in view of (I.62), converge to the function $-\log\delta_U(x,a)$. But then, [8], $-\log\delta_U(x,a)$ is a plurisubharmonic function on $U\times E$; and therefore, U is pseudoconvex.

A locally convex Hausdorff space is said to have the *approximation property* if for every compact subset K of E, every $\alpha \in cs(E)$ and every $\varepsilon > 0$ there exists a continuous linear map T from E into E such that $\dim_C T(E) < \infty$ and $\alpha(x-T(x)) < \varepsilon$ for all $x \in K$. M. Schottenholer has shown that in such space E, every open subset U of E which is polynomially convex is pseudoconvex and finitely Runge and conversely [18]. In particular,

THEOREM I.3.7. *Let E be a locally convex Hausdorff space with the approximation property. Let U be a domain in E; then the following are equivalent:*

(i) U is pseudoconvex and finitely Runge.

(ii) U is holomorphically convex and Runge.

(iii) U is polynomially convex.

THEOREM I.3.8. *Let E be a locally convex Hausdorff space with the approximation property. Let U be a domain in E, and let* $\{D_n\}_{n\in N}$ *be a nested increasing sequence of domains in E approximating U from the inside. Then, if each of the domains* D_n *in polynomially convex, then U is pseudoconvex and Runge.*

PROOF: Since D_n is polynomially convex, we obtain from Theorem I.3.7. that each D_n is pseudoconvex and finitely

Runge. But D_n being finitely Runge implies Runge [18]. Hence, each D_n is pseudoconvex and Runge. According to the Theorems I.3.5. and I.3.6., the limit of the given sequence is pseudoconvex and Runge.

CHAPTER II
KOBAYASHI AND CARATHÉODORY DISTANCES FOR COMPLEX BANACH MANIFOLDS

§1. Complex Analytic Banach Manifolds Over A Complex Banach Space

In this section we intend to extend the notion of a complex analytic manifold to a complex analytic Banach manifold over a complex Banach space. We need some definitions.

DEFINITION II.1.1. *Let* E *and* F *be two complex Banach spaces and let* U *be a non-empty open subset of* E. *A map* f *from* U *into* F *is said to be holomorphic in* U *if for every* $x_0 \in U$ *we can find a linear map* $L \in L(E,F)$ *such that*

$$\lim_{x \to x_0} \frac{\| f(x) - f(x_0) - L(x-x_0) \|}{\| x - x_0 \|} = 0 \quad .$$

Observe that the Goursat's Theorem for C^n extends in this case of the complex Banach spaces and therefore the above definition is equivalent to the definition given in Definition 0.10.

DEFINITION II.1.2. *Let* M *be a locally connected Hausdorff space and let* E *be a complex Banach space. Let* U *be a non-empty, open and connected subset of* M. *A map* ϕ *which is a homeomorphism from* U *onto some open subset of* E *is called a coordinate map and the pair* (U, ϕ) *is called a coordinate system.*

DEFINITION II.1.3. *Let M be a locally connected Hausdorff space and let E be a complex Banach space. A complex analytic structure on M is a collection F of coordinate systems* $\{(U_\alpha, \phi_\alpha): \alpha \in A\}$, *A some index set, satisfying the following three properties:*

(i) $M = \bigcup_{\alpha \in A} U_\alpha$.

(ii) The maps $\phi_\beta \circ \phi_\alpha^{-1}: \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ *are biholomorphic for all pairs* $\alpha, \beta \in A$.

(iii) The collection is maximal with respect to property (ii): that is, if (U, ϕ) *is a coordinate system such that* $\phi \circ \phi_\alpha^{-1}$ *and* $\phi_\alpha \circ \phi^{-1}$ *are holomorphic whenever they are defined for all* $\alpha \in A$ *then* $(U, \phi) \in F$.

DEFINITION II.1.4. *A complex analytic Banach manifold over a complex Banach space is a pair (M, F) consisting of a locally connected Hausdorff space M together with a complex analytic structure F.*

In the sequel a complex analytic Banach manifold (M,F) will be denoted by simply M. It is clear that if U is a non-empty open subset of a complex analytic Banach manifold M, then U itself is a complex analytic Banach manifold with complex analytic structure F_U given by $F_U = \{(U_\alpha \cap U, \phi_\alpha | U_\alpha \cap U): (U_\alpha, \phi_\alpha) \in F\}$.

DEFINITION II.1.5. *Let M be a complex analytic Banach manifold and let U be a non-empty open subset of M. A mapping f from U* *into C is said to be holomorphic in U if the mapping* $f \circ \phi^{-1}$ *is a holomorphic mapping for every coordinate map* ϕ *on M. In this case we write* $f \in H(U,C)$.

DEFINITION II.1.6. *Let M and N be two complex analytic Banach manifolds. A continuous mapping f from M into N is said to be holomorphic if* $g \circ f$ *is an holomorphic mapping on* $f^{-1}\{$*domain of g*$\}$ *for every* $g \in H(N,C)$. *In this case we write* $f \in H(M,N)$.

We note here that the above definition does not require the two manifolds M and N to be manifolds over the same Banach space. We are now proving a Theorem similar to the "analytic continuation Theorem" for complex analytic Banach manifolds.

THEOREM II.1.7. *Let M and N be two complex analytic Banach manifolds over the complex Banach spaces E and F respectively. Suppose that M is connected and that f and g are two holomorphic mappings from M into N. If there exists some non-empty and open subset U of M such that f = g on U, then f = g on M.*

PROOF: Let $S = \{x \in M\text{: } f = g \text{ on some neighborhood of } x \text{ in } M\}$. By hypothesis and the definition of the set S it follows that S is both an open and non-empty subset of M. We claim that the set S is closed too.

Since f = g on S and since both f and g [II.1.6.] are continuous we have that f = g on $\overline{S}$. Let $z \in \overline{S}$ and let $y = f(z) = g(z)$.

Let W_k be a neighborhood of y in N such that the coordinate map w_k is an homeomorphism onto some open subset P of F.

Let Q_{m_0} be a neighborhood of z in M such that the coordinate map q_{m_0} is an homeomorphism onto some open subset P_1 of E, and $Q_{m_0} \subset f^{-1}(W_k) \cap g^{-1}(W_k)$. Define

$$\text{(II.1)} \qquad Q_{m_k} = g^{-1}(W_k) \cap Q_{m_0} \cap f^{-1}(W_k)$$

where here by W_k, we denote the subset of W_k where both $f^{-1}(W_k)$ and $g^{-1}(W_k)$ are defined. Since by its definition Q_{m_k} contains z we obtain that the set Q_{m_k} is non-empty and moreover it is clear that is open. Define two maps t_1 and t_2 on $q_{m_0}(S \cap Q_{m_k})$ by

$$t_1 = w_k \circ f \circ q_{m_0}^{-1} \quad \text{and} \quad t_2 = w_k \circ g \circ q_{m_0}^{-1} .$$

From the definitions of S and Q_{m_k} it follows that we can find a $b \in q_{m_0}(S \cap Q_{m_k})$ and then because of (II.1) we obtain $t_1(b) = t_2(b)$. Since b is an arbitrary element we obtain

$$\text{(II.2)} \qquad t_1 = t_2 \text{ on } q_{m_0}(S \cap Q_{m_k}).$$

But clearly $q_{m_0}(S \cap Q_{m_k}) \subset q_{m_0}(Q_{m_k})$ and then in view of the (II.2) and of the Theorem 0.14., we infer that $t_1 = t_2$ on $q_{m_0}(Q_{m_k})$, or $w_k \circ f \circ q_{m_0}^{-1} = w_k \circ g \circ q_{m_0}^{-1}$ on $q_{m_0}(Q_{m_k})$. But both mappings w_k and q_{m_0} are homeomorphisms and therefore the later equality in view of that fact yields

$$\text{(II.3)} \qquad f = g \text{ on } Q_{m_k} .$$

But (II.3) implies that the set Q_{m_k} is a subset of the set S. But from the construction of the set Q_{m_k} we have that $z \in Q_{m_k}$ and hence we have that $z \in S$. Therefore we have shown that $\overline{S} \subset S$ and hence we obtain that the set S is closed. Finally, since M is assumed to be connected and since we have shown that S is both open and closed we obtain $S = M$ and the Theorem is proved.

§2. The Kobayashi Pseudodistance On a Complex Analytic Banach Manifold

In the sequel by Δ we will denote the open unit disk in the complex plane and by $\partial\Delta$ its boundary. If M is a complex analytic Banach manifold by $H(\Delta,M)$, we will denote the set of all holomorphic mappings from Δ into M. We will denote by ρ the Poincaré-Bergman distance on the unit disk Δ. Recall that ρ is defined for every pair of points z and z' of Δ by

$$\tanh(\tfrac{1}{2}\rho(z,z')) = \frac{|z - z'|}{|1 - \bar{z}z'|} \quad .$$

DEFINITION II.2.1. *Let M be a complex analytic manifold. Let x and y be two points of M. Choose n + 1 points* $p_0, p_1, p_2, \ldots, p_n$ *of M, such that* $p_0 = x$ *and* $p_n = y$. *Choose 2n points* $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ *of* Δ. *Finally, choose n mappings* $f_1, f_2, \ldots, f_n$ *of* $H(\Delta,M)$ *such that for each* $i = 1,2,\ldots,n$ *we have*

(II.4) $\qquad f_i(a_i) = p_{i-1}$ *and* $f_i(b_i) = p_i$.

For each choice of the above sets of points $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$, $\{p_i\}_{i=0}^n$ *and mappings* $\{f_i\}_{i=1}^n$, *thus made, we consider the sum*

$$\text{(II.5)} \qquad \sum_{i=1}^{n} \rho(a_i, b_i) \ .$$

We denote by $d_M(x,y)$ *the infimum, taken over all possible choices of the points* $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n, \{p_i\}_{i=0}^n$ *and mappings* $\{f_i\}_{i=1}^n$ *satisfying (II.4), of the numbers obtained in (II.5).*

The function obtained from the above definition has the following properties.

THEOREM II.2.2. *Let M be a complex analytic Banach manifold. The function* d_M: $M \times M \to R$ *defined by* $(x,y) \to d_M(x,y)$ *is a pseudodistance on M.*

PROOF: Consider two points x and y in M. If x = y, we may choose points p_0 and p_1 such that $p_0 = x$ and $p_1 = y$. Let $f \in H(\Delta, M)$ and points $a_1, b_1 \in \Delta$ such that $f(a_1) = p_0$ and $f(b_1) = p_1$. Since $p_0 = p_1$, we can take $a_1 = b_1$. But then $\rho(a_1, b_1) = 0$ for that choice of a_1, b_1, p_0, p_1 and f. Since d_M is always greater than or equal to zero and since $d_M(x,x)$ is the infimum of the numbers (II.5) of the Definition II.2.1., that infimum in this case must be equal to zero. Therefore, we have shown that for each x in M we have $d_M(x,x) = 0$. The triangle inequality is satisfied trivially and hence the Theorem is proved.

THEOREM II.2.3. *Let M be a complex analytic Banach manifold. The function* d_M: $M \times M \to R$ *defined by* $(x,y) \to d_M(x,y)$ *is continuous.*

PROOF: Let r be any positive real number. It suffices to show that the sets

$$A = \{(x,y): d_M(x,y) < r\}$$
$$B = \{(x,y): d_M(x,y) > r\}$$

are open in $M\times M$.

We first prove that A is open. Let (α,β) be any point of A. Then $d_M(\alpha,\beta) = q < r$. Take $p = \frac{1}{2}(r-q)$ and consider the open set U in $M\times M$ defined by

$$U = B_p(\alpha) \times B_p(\beta).$$

If $(x,y) \in U$, then we have that

$$d_M(x,y) \leq d_M(x,\alpha) + d_M(\alpha,\beta) + d_M(\beta,y) < p+q+r = q+2p = r.$$

The last inequality implies that $(\alpha,\beta) \in U \subset A$; and therefore, A is open.

Now, let (α,β) be any point of B. Then we have that $d_M(\alpha,\beta) > r$. Take $p = \frac{1}{2}(q-r)$ and consider the open set V in $M\times M$ defined by

$$V = B_p(\alpha) \times B_p(\beta).$$

If $(x,y) \in V$ then we have that

$$d_M(\alpha,\beta) \leq d_M(\alpha,x) + d_M(x,y) + d_M(y,\beta).$$

The above inequality implies that

$$d_M(x,y) \geq d_M(\alpha,\beta) - d_M(\alpha,x) - d_M(\beta,y) > q-p-p = q-2p = r.$$

The above inequality implies that $(\alpha,\beta) \in V \subset B$; and therefore, the set B is open.

The pseudodistance d_M defined in the Definition II.2.1. on a complex analytic Banach manifold is called the Kobayashi pseudodistance on M. This pseudodistance has the property, as we will see below, that it is the largest pseudodistance defined on M for which every holomorphic mapping from Δ into M is distance decreasing. Moreover, if M and N are two complex analytic Banach manifolds furnished with the Kobayashi pseudodistances d_M and d_N respectively, then every biholomorphic mapping from M onto N is an isometry.

THEOREM II.2.4. *Let M and N be two complex analytic Banach manifolds. If f is a holomorphic mapping from M into N, then for each pair* $x,y \in M$ *we have*

$$d_M(x,y) \geq d_N(f(x),f(y)).$$

PROOF: It is enough to observe that each choice of points $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n, \{p_i\}_{i=0}^n$ and holomorphic mappings $\{f_i\}_{i=1}^n$ made for defining d_M, yields a choice of points $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$, $\{f(p_i)\}_{i=0}^n$ and holomorphic mappings $\{f\circ f_i\}_{i=1}^n$ needed for generating d_N.

THEOREM II.2.5. *Let M and N be two complex analytic Banach manifolds. If f is a biholomorphic mapping from M onto N, then for each pair* $x,y \in M$ *we have that*

$$d_M(x,y) = d_N(f(x),f(y)).$$

The proof of the above theorem follows immediately from the Theorem II.2.4. We are proving

THEOREM II.2.6. *Let M be a complex analytic Banach manifold and let d' be any pseudodistance on M such that for every pair $\alpha, \beta \in \Delta$ and for every $f \in H(\Delta, M)$ we have*

$$d'(f(\alpha), f(\beta)) \leq \rho(\alpha, \beta).$$

Then we have that

$$d_M(x, y) \geq d'(x, y) \text{ for } x, y \in M.$$

PROOF: Let $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n, \{p_i\}_{i=0}^n$ and $\{f_i\}_{i=1}^n$ be as in Definition II.2.1. Then

$$\begin{aligned} d'(x,y) &\leq \sum_{i=1}^{n} d'(p_{i-1}, p_i) = \sum_{i=1}^{n} d'(f_i(a_i), f_i(b_i)) \\ &\leq \sum_{i=1}^{n} \rho(a_i, b_i). \end{aligned}$$

But, the infimum of the right hand side of the above inequality taken over all possible choices of the points $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n, \{p_j\}_{j=0}^n$ and mappings $\{f_i\}_{i=1}^n$ is by definition equal to $d_M(x,y)$. Hence, the above inequality proves the Theorem.

THEOREM II.2.7. *Let M and N be two complex analytic Banach manifolds. Then for every pair $x, y \in M$ and for every pair $x', y' \in N$ we have*

$$\begin{aligned} \text{(II.6)} \quad d_M(x, y) + d_N(x', y') &\geq d_{M\times N}((x, x'), (y, y')) \\ &\geq \max(d_M(x, y), d_N(x', y')). \end{aligned}$$

PROOF: Define a mapping f from M into M×N by $f(x) = (x,x')$, and a mapping g from N into M×N by $g(x') = (y,x')$. Theorem II.2.4. implies that both mappings f and g are distance-decreasing, and hence we have that

$$d_M(x,y) + d_N(x',y') \geq d_{M\times N}(f(x),f(y)) + d_{M\times N}(g(x'),g(y')) =$$

$$= d_{M\times N}((x,x'),(y,x')) + d_{M\times N}((y,x'),(y,y')) \geq d_{M\times N}((x,x'),(y,y')).$$

From the above inequalities we obtain

$$\text{(II.7)} \quad d_M(x,y) + d_N(x',y') \geq d_{M\times N}((x,x'),(y,y')).$$

Define a new mapping K from M×N into M by the rule $K(x,y) = x$, and a mapping k from M×N into N by the rule $k(x,y) = y$. Appealing to the Theorem II.2.4., once again we obtain

$$\text{(II.8)} \quad d_{M\times N}((x,y),(x',y')) \geq d_M(K(x,y),K(x',y')) = d_M(x,x')$$

and

$$\text{(II.9)} \quad d_{M\times N}((x,y),(x',y')) \geq d_N(k(x,y),k(x',y')) = d_N(y,y').$$

Finally, combining inequalities (II.7), (II.8) and (II.9) we obtain (II.6).

The inequality $d_{M\times N}((x,y),(x',y')) \geq \max(d_M(x,y),d_N(x',y'))$ can actually become an equality. In particular, we obtain such an example by taking $M = N = \Delta$. Also the equality holds if Δ is substituted by $\Delta^m = \Delta\times\Delta\times\ldots\times\Delta$. The last equality also serves as an example to the fact that the Kobayashi pseudodistance does not coincide with the Bergman metric on Δ^m unless $m = 1$.

DEFINITION II.2.8. *Let M be a complex analytic Banach manifold and let X be a connected and locally pathwise connected topological space. We will say that X is a covering manifold of M, if and only if, there is a continuous mapping π from X onto M with the property that each point y of M has a neighborhood V whose inverse image under π is a disjoint union of open sets in X each homeomorphic with V under π.*

THEOREM II.2.9. *Let M be a complex analytic Banach manifold and let X be a covering manifold of M with covering projection π. Let $p,q \in M$ and $x,y \in X$ be such that $\pi(x) = p$ and $\pi(y) = q$. Then*

$$d_M(p,q) = \inf_{y \in X} \{d_X(x,y): \pi(y) = q\}.$$

PROOF: Appealing to the Theorem II.2.4., we obtain

$$d_M(p,q) \leq \inf_{y \in X} d_X(x,y).$$

Assuming that strict inequality holds, we can find some positive real number ε such that

$$d_M(p,q) + \varepsilon < \inf_{y \in X} d_X(x,y). \tag{II.10}$$

But by the definition of d_M, it follows that there exist points $a_1,a_2,\ldots,a_k,b_1,b_2,\ldots,b_k$ of the unit disk Δ, points $p_0,p_1,\ldots,p_k$ of M, and holomorphic mappings $f_1,f_2,\ldots,f_k$ of Δ into M such that

$$p = f_1(a_1),\ f_2(b_1) = f_2(a_2),\ldots,\ f_{k-1}(b_{k-1}) = f_k(a_k),\ f_k(b_k) = q$$

and

$$d_M(p,q) + \varepsilon > \sum_{i=1}^{k} \rho(a_i,b_i).$$

Now, we can lift the holomorphic mappings $f_1, f_2, \ldots, f_k$ to holomorphic mappings $F_1, F_2, \ldots, F_k$ of Δ into X [19] in such a way that

$$x = F_1(a_1)$$

$$F_i(b_i) = F_{i+1}(a_{i+1}) \quad \text{for each } i = 1,2,\ldots,k-1$$

$$\pi \circ F_i = f_i \qquad \text{for each } i = 1,2,\ldots,k .$$

By letting $y = F_k(b_k)$, then $\pi(y) = q$ and $d_X(x,y) \le \sum_{i=1}^{k} \rho(a_i,b_i)$. Hence,

$$d_X(x,y) < d_M(p,q) + \varepsilon;$$

and then the above found inequality contradicts (II.10).

It is not known whether the infimum is attained for some y in X even for the case of the complex manifolds.

§3. The Carathéodory Pseudodistance On a Complex Analytic Banach Manifold

In the sequel by $H(M)$ we will denote the set of all holomorphic mappings from a complex analytic Banach manifold M into the unit disk Δ of the complex plane. Also, by ρ we will denote, as previously, the Poincaré-Bergman metric on Δ.

DEFINITION II.3.1. *Let M be a complex analytic Banach manifold and let x and y be two points of M. We will denote by*

$c_M(x,y)$ *the real number defined as the supremum of the numbers* $\rho(f(x),f(y))$ *taken with respect to the family of the mappings belonging to* $H(M)$. *In other words, we define*

$$c_M(x,y) = \sup_{f \in H(M)} \{\rho(f(x),f(y))\}.$$

It is easy to see that

THEOREM II.3.2. *Let* M *be a complex analytic Banach manifold. The function* $c_M\colon M\times M \to R$ *defined by* $(x,y) \to c_M(x,y)$ *is continuous and a pseudodistance on* M.

The above defined pseudodistance is called the Carathéodory pseudodistance on M. From what it follows, we can see that the Carathéodory pseudodistance shares many properties with the Kobayashi pseudodistance, and in particular, the d_M is greater than or equal to c_M. Moreover, the Carathéodory pseudodistance may also be considered as a generalization of the Poincaré-Bergman metric for Δ. We return now to the properties of c_M.

THEOREM II.3.3. *Let* M *be a complex analytic Banach manifold. For every pair of points* x *and* y *of* M *we have*

$$d_M(x,y) \geq c_M(x,y).$$

PROOF: Choose points $p_0,p_1,\ldots,p_n$ of M, points $a_1,a_2,\ldots,a_n$, $b_1,b_2,\ldots,b_n$ of Δ, and mappings $f_1,f_2,\ldots,f_n$ of $H(\Delta,M)$ such that for each $i = 1,2,\ldots,n$ we have

$$f_i(a_i) = p_{i-1} \text{ and } f_i(b_i) = p_i.$$

Let F be a holomorphic mapping from M into Δ. Then the mappings $F \circ f_i$ are holomorphic mappings from Δ into Δ for each $i = 1,2,\ldots,n$. Schwarz's lemma implies that all these mappings are distance decreasing with respect to the Poincare-Bergman metric, and hence, for each $i = 1,2,\ldots,n$, we have

$$\rho(a_i,b_i) \geq \rho(F \circ f_i(a_i), F \circ f_i(b_i)).$$

Therefore,

$$\sum_{i=1}^{n} \rho(a_i,b_i) \geq \sum_{i=1}^{n} \rho(F \circ f_i(a_i), F \circ f_i(b_i)) \geq \rho(F \circ f_1(a_1), F \circ f_n(b_n)) =$$

$$= \rho(F(x),F(y)).$$

Finally, in view of the above inequality, we obtain

$$d_M(x,y) = \inf \sum_{i=1}^{n} \rho(a_i,b_i) \geq \sup \rho(F(x),F(y)) = c_M(x,y).$$

The proofs of the following Theorems are similar to the ones in the previous section and therefore will be omitted.

THEOREM II.3.4. *Let M and N be two complex analytic Banach manifolds. Let f be a holomorphic mapping from M into N. Then*

$$c_M(x,y) \geq c_N(f(x),f(y))$$

for all pairs of points x and y of M.

THEOREM II.3.5. *Let M and N be two complex analytic Banach manifolds. Then every biholomorphic mapping f from M onto N is an isometry, i.e.,*

$$c_M(x,y) = c_N(f(x),f(y)).$$

The following Theorem indicates that the Carathéodory pseudodistance c_M is the smallest pseudodistance defined on a complex analytic Banach manifold for which every holomorphic mapping f from M into Δ is distance decreasing.

THEOREM II.3.6. *Let M be a complex analytic Banach manifold. Let d be any pseudodistance defined on M such that for every pair of points x and y of M we have*

$$d(x,y) \geq \rho(f(x),f(y))$$

for every holomorphic mapping f from M into Δ. Then for every pair of points x and y of M we have

$$c_M(x,y) \leq d(x,y).$$

THEOREM II.3.7. *Let M and N be two analytic Banach manifolds. Then for every pair $x,y \in M$ and $x',y' \in N$, we have*

$$c_M(x,y) + c_N(x',y') \geq c_{M\times N}((x,x'),(y,y')) \geq \max(c_M(x,y), c_N(x',y')).$$

We are closing this section with the remark that in the case of the k-dimensional polydisk Δ^k, we have that the Kobayashi and Carathéodory pseudodistances agree [13].

BIBLIOGRAPHY

[1] Aurich, V., *Characterization of Domains of Holomorphy Over an Arbitrary Product of Complex Lines, Diplomarbeit, München, 1973.*

[2] Bers, L., *Introduction to Several Complex Variables, Lectures, Courant Institute of Mathematical Sciences, N.Y.U., N.Y., 1964.*

[3] Chen, S. S., *Carathéodory Distance and Convexity With Respect to Bounded Holomorphic Functions, Proceedings of* A.M.S., *V. 39, pp. 305-307, 1973.*

[4] Coeuré, G., *Analytic Functions and Manifolds in Infinite Dimensional Spaces, Notas de Matemática (52), North Holland, 1977.*

[5] Dineen, S., *Bounding Subsets of a Banach Space, Math. Ann. 192, 1971.*

[6] Dunford, N. and Schwartz, J., *Linear Operators, Part I, General Theory, Interscience, N.Y., 1957.*

[7] Fuks, B. A., *Analytic Functions of Several Complex Variables, Translations of Mathematical monographs,* A.M.S., *V. 13, 1965.*

[8] Fuks, B. A., *Analytic Functions of Several Complex Variables, Translations of Mathematical monographs,* A.M.S., *V. 14, 1965.*

[9] Gruman, L., *The Levi Problem in Certain Infinite Dimensional Vector Spaces, Ill. J. Math. 18, pp. 20-26, 1974.*

[10] Gunning, R. C., and Rossi, H., *Analytic Functions of Several Complex Variables, Prentice-Hall, Englewood Cliffs, N.J., 1965.*

[11] Josefson, B., *A Counterexample to the Levi Problem, "Proceedings on Infinite Dimensional Holomorphy", Springer Lecture Notes 264, pp. 168-177, 1974.*

[12] Kim, D. S., *Carathéodory Distance and Bounded Holomorphic Functions, Duke Mathematical Journal, V. 41, pp. 333-338, 1974.*

[13] Kobayashi, S., *Hyperbolic Manifolds and Holomorphic Mappings, Marcel Dekker, N.Y., 1970.*

[14] Matos, M., *Domains of τ-Holomorphy in a Separable Banach Space, Math. Ann. 195, pp. 273-277, 1972.*

[15] Nachbin, L., *Holomorphic Functions, Domains of Holomorphy, Local Properties, North Holland, 1970.*

[16] Nachbin, L., *Concerning Spaces of Holomorphic Mappings, Seminar Lectures Rutgers University, N.J., 1970.*

[17] Narashimhan, R., *Several Complex Variables, University of Chicago Press, Chicago, 1971.*

[18] Schottenhloher, M., *Polynomial Approximations on Compact Sets, Infinite Dimensional Holomorphy and Applications, North Holland, 1977.*

[19] Warner, W. F., *Foundations of Differentiable Manifolds and Lie Groups, Scott and Foresman and Co., Glenview, Ill., 1971.*

[20] ______________, *Infinite Dimensional Holomorphy and Applications, International Symposium, Notas de Matemática, North Holland, 1977.*

BIOGRAPHICAL SKETCH

Panos E. Livadas was born on February 22, 1944, in Athens, Greece, to Evangelos and Mary Livadas. He attended private schools until he received his high school diploma from "Parthenon" high school of Athens. He attended Aristotle University of Thessaloniki in Greece where he graduated Magna Cum Laude from the school of Arts and Sciences and received his Bachelor of Science in mathematics with a minor in physics in February of 1970.

He began his graduate studies in March of 1970 at Georgia Southern College, in Stateboro, Georgia; and he received his Master of Science degree in mathematics in August of 1971. In December of 1971, he married the former Debra Anne Waters; and in September of 1972, he entered the University of Florida where he had been awarded a teaching assistantship and would further his studies toward the Ph.D. degree. He was introduced to the Theory of Several Complex Variables by Dr. D. S. Kim who served as his advisor until the year 1976 when Dr. Kim's association with the University of Florida ended. Since that time his new advisor has been Dr. S. Chen.

The past two years the author was employed by the University of North Florida where he served the first year as an instructor and the second as instructor and academic advisor during which time Dr. S. Chen was on sabattical.

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